A Lie group analogue of the coset poset of abelian subgroups

Bilkent Topology Seminar, Nov 30 2020

G group, $\mathcal{F}$ family of subgroups of G

$(\mathcal{F}, G) \rightsquigarrow$ simplicial complex $\mathcal{C}(\mathcal{F}, G)$ whose n-simplices are the n-chains of cosets

$$g_0 A_0 \subset \cdots \subset g_n A_n \qquad g_i \in G \quad , \ A_i \in \mathcal{F}$$

$|\mathcal{C}(\mathcal{F}, G)|$ order complex associated to the "coset poset" with respect to $\mathcal{F}$.

→ Abels and Holz studied the relationship between higher connectivity of $|\mathcal{C}(\mathcal{F}, G)|$ and the algebraic properties of G with respect to $\mathcal{F}$:

$$|\mathcal{C}(\mathcal{F}, G)| \text{ connected} \iff G = \langle \bigcup_{A \in \mathcal{F}} A \rangle$$

$|\mathcal{C}(\mathcal{F}, G)|$ 1-connected $\iff G \cong$ amalgamated product $A \in \mathcal{F}$ along its intersections

$\rightsquigarrow$ let $\mathcal{F} = Ab :=$ abelian subgroups

C. Okay noted that $|\mathcal{C}(Ab, G)|$ 1-connected $\iff G$ is abelian (more over $|\mathcal{C}(Ab, G)| \simeq *$)

Goal: Provide an analogue of $|\mathcal{C}(Ab, G)|$ and Okay's result for a compact Lie group G.

$\rightsquigarrow$ Adem, F. Cohen and Torres-Giese studied a filtration of the classifying space BG for Lie groups or discrete groups $\cdots \subset B(q,G) \subset \cdots \subset B(2,G) \subset BG$

e.g. $\quad B(2,G) := |k \mapsto \mathrm{Hom}(\mathbb{Z}^k, G) \subset G^k|$

$\rightsquigarrow$ G discrete $\quad B(2,G) = \bigcup_{A \in Ab} BA$

after looping $\quad \Omega B(2,G) \simeq \Omega E(2,G) \times G \quad$ (G conn)

$E(2,G)$ is the homotopy fiber
$$\begin{tikzcd} E(2,G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & EG \arrow[d] \\ B(2,G) \arrow[r] & BG \end{tikzcd}$$

Thus a simplicial model for $E(2,G)$ can be given as follows

For each n, let

$E_n(2,G) = \{ (g_0, \dots, g_n) \in G^n : \langle g_0^{-1}g_1, \dots, g_{n-1}^{-1}g_n \rangle \in Ab \}$

When G is discrete, $E(2,G) \simeq |\mathcal{C}(Ab, G)|$

(Adem–Holz)

$E(2,G)$ is a good analogue for $|\mathcal{C}(Ab, G)|$ when G is a Lie group.

Thm: (Antolín-Camarena, Gritschacher, V) Let G compact Lie group, then

G is a torus $\iff \pi_i(E(2,G)) = 0$ for $i = 1, 2$ and 4.

* Moreover G is abelian $\iff E(2,G)$ is contractible

A predecessor: $E(2,G)_1 := |G \times \mathrm{Hom}(\mathbb{Z}^k, G)_1|$ conn comp of $(1,\dots,1)$

Thm: (Adem–Gómez): G comp Lie grp.

G is abelian $\iff E(2,G)_1$ is rationally acyclic

Proof idea: $H^*(E(2,G)_1; \mathbb{Q}) \cong H^*(G/T \times G/T)^W = 0$

$\Rightarrow G/T$ is rationally acyclic $\iff G = T$

The commutator map $c : E(2,G) \to B[G,G]$

Lemma: Let $g,h,k \in G$ s.t. $\langle g^{-1}h,\ h^{-1}k\rangle \in Ab$. Then

$$[g,h][h,k] = [g,k]$$

Def: For every n, $c_n : E_n(2,G) \to [G,G]^n = B_n[G,G]$

$$(g_0,\dots,g_n) \longmapsto ([g_0,g_1],\dots,[g_{n-1},g_n])$$

which induces a simplicial map $c_\bullet : E_\bullet(2,G) \to B_\bullet[G,G]$

$$|c_\bullet| =: c : E(2,G) \to B[G,G]$$

Prop: Let G compact Lie group or a discrete group

① c null-homotopic $\Rightarrow$ G is homotopy abelian
(i.e., $c : G\times G \to G$ $(x,y) \mapsto [x,y]$ is null)

② $c_* : \pi_1 E(2,G) \to \pi_0[G,G]$ is surjective

Proof idea: ① There is a commutative diagram

$$\begin{tikzcd}
E(2,G) \arrow[r, "c"] & B[G,G] \\
G * G \arrow[u] \arrow[r, "c'"] \arrow[d, phantom, "\cong" rotate=90] & \Sigma[G,G] \arrow[u] \arrow[d, phantom, "="] \\
\Sigma G\wedge G \arrow[r, "\Sigma\tilde{c}"] & \Sigma G
\end{tikzcd}$$

(in thick geometric realization)

$$\begin{tikzcd}
G\times G \arrow[r, "c"] \arrow[d] & G \\
G\wedge G \arrow[ur, "\tilde{c}"'] &
\end{tikzcd}$$

$c \simeq ct \Rightarrow \Sigma G\wedge G \to B[G,G]$ is null

$2 \Rightarrow G\wedge G \xrightarrow{\tilde{c}} [G,G]$ is null

② G discrete $\qquad c_*: \pi_1 E(2,G) \to [G,G]$

$$\{x_{g,h} : x_{g,1} = 1,\ x_{g,h}\cdot x_{h,k} = x_{g,k} \text{ if } \langle g^{-1}h, h^{-1}k\rangle \in Ab\}$$

$c_*(x_{g,h}) = [g,h]$

G compact

$$\begin{tikzcd} E(2,G^\delta) \arrow[r, "c^\delta"] \arrow[d] & B[G^\delta,G^\delta] \arrow[d] \\ E(2,G) \arrow[r, "c"'] & B[G,G] \end{tikzcd}$$

then in $\pi_1(-)$: $\qquad \pi_1 E(2,G^\delta) \overset{c_*^\delta}{\twoheadrightarrow} \pi_0[G^\delta,G^\delta] \to \pi_0[G,G]$ $\quad$ surj on path-conn comp

$\Rightarrow$ c_* is surjective as well. $\square$

Rmks: There are htpy abelian compact Lie groups that are not abelian, e.g. $G = S^1 \times_{\mathbb{Z}/2} Q_8$ central prod

Homotopy abelian compact Lie groups can be classified as follows:

G is htpy abelian $\Leftrightarrow$ G is a central extension of $\pi_0(G)$ by a torus.

Prop: Let G comp Lie group. If $\pi_4 E(2,G) = 0$ then G_0 is a torus.

Proof idea: It suffices to show $[G,G]_0$ is abelian.

If it is not can find $SU(2) \hookrightarrow [G,G]_0$ inducing iso in $\pi_3(-)$.

The map $\varepsilon: E(2, SU(2)) \to BSU(2)$ has a homotopy section s after looping ε

$$
\begin{array}{ccc}
\pi_3 \Omega E(2, SU(2)) & \twoheadrightarrow & \pi_3 SU(2) \\
\downarrow & & \downarrow \\
\pi_3 \Omega E(2, G) & \to & \pi_3 [G,G]_0
\end{array}
$$

thus $\pi_4(E(2,G))$ can't be zero.

$\square$

Prop: Let G compact Lie group, and suppose G_0 is a torus. Then if $E(2,G)$ is 2-connected, ε is null-homotopic.

Sketch Proof: $E(2,G)$ 1-conn $\Rightarrow [G,G] \subset G_0$, thus a torus. Then $\varepsilon \in H^2(E(2,G); \pi_1[G,G]) = 0$

$\Rightarrow \varepsilon$ is null.

$\square$

Sketch proof main thm: Suppose $\pi_i(E(2,G)) = 0$ $i = 1,2,4$

$\pi_4(E(2,G)) = 0 \Rightarrow G_0$ a torus

$\pi_2(E(2,G)) = 0 \Rightarrow G$ is htpy abelian

$\overset{2}{\Rightarrow} G$ central extension $\pi_0(G)$ by a torus

$\pi_1 E(2,G) = 0 \Rightarrow \underbrace{[G,G]}_{\text{connected}} \subset G_0 = T$

As G is a central extension by G_0, and $\pi_0(G)$ is finite, $[G,G]$ is finitely generated. Thus $[G,G]=1$. $\square$

$c : E(2, O(2)) \to BSO(2)$ also splits after looping.

<u>Question</u>: If G compact Lie group, does

$\Omega c : \Omega E(2,G) \to [G,G]$ has a section? (up to htpy)