Postulates of quantum mechanics

In this course we will be concerned with finite-dimensional quantum mechanics.

Postulate 1 - (Pure) states

A state is a complete description of a physical system and it is specified by a line (ray) in the Hilbert space.

Let V be a Hilbert space

$$P(V) = \{ v \in V \mid v \neq 0 \} \big/ \; v' \sim v \text{ if } v' = \lambda v \text{ for some non-zero } \lambda \in \mathbb{C}.$$

Alternatively

$$P(V) = \{ v \in V \mid \|v\| = 1 \} \big/ \; v' \sim v \text{ if } v' = \lambda v \text{ for some } \lambda \in U(1)$$

Notation $[v]$ equivalence class.

Ex - Qubits (Spin - $\frac{1}{2}$)

Hilbert space $= \mathbb{C}^2$ with orthonormal basis $\{|0\rangle, |1\rangle\}$

A unit vector has the form

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

where $|a|^2 + |b|^2 = 1$

Notation $|a| = \sqrt{a^* a}$.

Reparametrization:

$a = r_0 e^{i\alpha_0}$ $\quad r_0, r_1 \in \mathbb{R}_{\geq 0}$

$b = r_1 e^{i\alpha_1}$ $\quad 0 \leq \alpha_0, \alpha_1 < 2\pi$

$$|\psi\rangle = r_0 e^{i\alpha_0}|0\rangle + r_1 e^{i\alpha_1}|1\rangle$$

$$= e^{i\alpha_0}\left(r_0|0\rangle + r_1 e^{i(\alpha_1 - \alpha_0)}|1\rangle\right)$$

global phase $\in U(1)$

$$\sim r_0|0\rangle + r_1 e^{i\varphi}|1\rangle \qquad \varphi = \alpha_1 - \alpha_0$$

$$|a|^2 + |b|^2 = 1 \implies r_0^2 + r_1^2 = 1$$

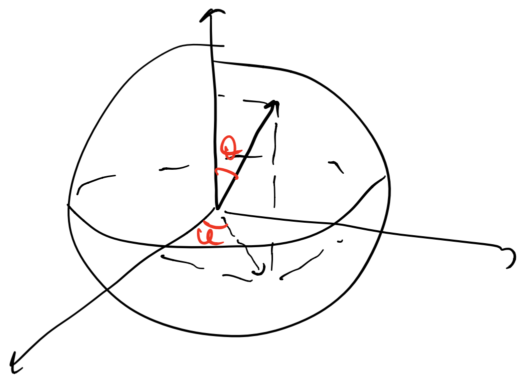
$$|\psi\rangle \sim \cos\frac{\theta}{2}|0\rangle + \left(\sin\frac{\theta}{2}\right)e^{i\varphi}|1\rangle$$

$$0 \leq \theta < \pi \qquad 0 \leq \varphi < 2\pi$$

The state space

$$P(\mathbb{C}^2) = \{ |\psi\rangle \mid \|\psi\| = 1 \} / \sim$$

is a sphere, called the Bloch sphere



$0 \leq \theta < \pi$

$0 \leq \varphi < 2\pi$

States as linear operators

Using outer products we can define

$$\begin{aligned} P(V) &\hookrightarrow \operatorname{Proj}(V) \\ [\psi] &\longmapsto |\psi\rangle\langle\psi| \end{aligned}$$

$$\left.\begin{array}{l} \psi' = \lambda\psi \\ \lambda \in U(1) \end{array}\right\} \quad \psi' \mapsto |\psi'\rangle\langle\psi'|$$

$$\begin{aligned} &= \lambda|\psi\rangle\langle\psi|\lambda^* \\ &= \lambda\lambda^* |\psi\rangle\langle\psi| \\ &= |\psi\rangle\langle\psi| \end{aligned}$$

Then consider

$$P(V) \hookrightarrow \operatorname{Proj}(V) \hookrightarrow \operatorname{Pos}(V) \hookrightarrow \operatorname{Herm}(V)$$

Ex - Qubits

$$\mathrm{Herm}(\mathbb{C}^2) = \{ A \in L(\mathbb{C}^2) \mid \boxed{A^\dagger = A} \}$$

$$A = \begin{pmatrix} r_1 & a \\ a^* & r_2 \end{pmatrix} \quad r_1, r_2 \in \mathbb{R}, \quad a \in \mathbb{C}$$

let $a = s_1 + i s_2 \qquad s_1, s_2 \in \mathbb{R}$

$\mathrm{Tr}\, A = r_1 + r_2 = \alpha_0$

$\det A = r_1 r_2 - (s_1^2 + s_2^2)$

$= \frac{1}{4}(\alpha_0^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2)$

$$A = \begin{pmatrix} r_1 & s_1 + i s_2 \\ s_1 - i s_2 & r_2 \end{pmatrix} =$$

$$\frac{1}{2}\left[\underbrace{(r_1 + r_2)}_{\alpha_0} I + \underbrace{2 s_1}_{\alpha_1} X \underbrace{- 2 s_2}_{\alpha_2} Y + \underbrace{(r_1 - r_2)}_{\alpha_3} Z \right]$$

where I, X, Y, Z Pauli matrices.

Fact 1) $I, X, Y, Z \in \mathrm{Herm}(\mathbb{C}^2)$

2) Writing $\{ \sigma_i \}_{i=0}^{3}$ for Pauli's,

$$\boxed{\mathrm{Tr}\left(\frac{\sigma_i}{\sqrt{2}} \frac{\sigma_j}{\sqrt{2}} \right) = \delta_{ij}}$$

$$\mathrm{Tr}(\sigma_i) = \begin{cases} 2 & i = 0 \\ 0 & i \neq 0 \end{cases}$$

$\sigma_3 = i \sigma_1 \sigma_2$

basis therefor (real) vector space

$\mathrm{Herm}(\mathbb{C}^2)$.

So in summary $A \in \mathrm{Herm}(\mathbb{C}^2)$ can be written as

$$A = \frac{1}{2}\left[\alpha_0 I + \alpha_1 X + \alpha_2 Y + \alpha_3 Z \right) \quad \alpha_i \in \mathbb{R}.$$

<u>Exercise</u> Show that $A \in P(\mathbb{C}^2) \subset \mathrm{Herm}(\mathbb{C}^2)$

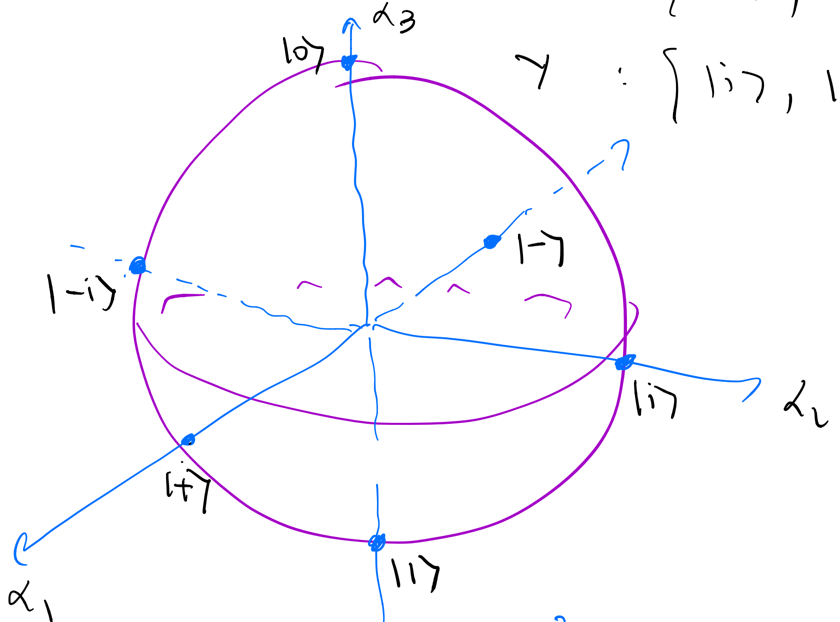
$\iff \alpha_0 = 1$ and $\boxed{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1}$

$P(\mathbb{C}^2)$ is the Bloch sphere.

<u>Ex</u> Eigenstates of Z : $\{ |0\rangle, |1\rangle \}$ $\quad \{ |0\rangle\langle 0|, |1\rangle\langle 1| \}$

X : $\{ |+\rangle, |-\rangle \}$ $\quad \{ |+\rangle\langle +|, |-\rangle\langle -| \}$

Y : $\{ |i\rangle, |-i\rangle \}$ $\quad \{ |i\rangle\langle i|, |-i\rangle\langle -i| \}$



$$|0\rangle\langle 0| = \frac{1}{2}\left(\sum_{i=0}^{3} \alpha_i \sigma_i \right) = \frac{1}{2}(I + Z)$$

$$|1\rangle\langle 1| = \frac{1}{2}(I - Z)$$

$$|+\rangle\langle +| = \frac{1}{2}(I + X)$$

$$|-\rangle\langle -| = \frac{1}{2}(I - X)$$

$$|\pm i\rangle\langle \pm i| = \frac{I \pm Y}{2}$$

Postulate 2 - Evolution

Time evolution of a quantum state is described by a unitary operator

$$|\psi'\rangle = U\,|\psi\rangle$$

where $|\psi\rangle$ is state at time t_1

$|\psi'\rangle$ is state at time t_2

and $U \in U(V)$ depends on t_1 and t_2.

Ex - Qubit

$U(\mathbb{C}^2)$ can be understood using the exponential map

$$\exp : \mathrm{Herm}(\mathbb{C}^2) \twoheadrightarrow U(\mathbb{C}^2)$$
$$A \mapsto e^{-iA}$$

(can be generalized to $\mathbb{C}^n$)

Writing $A = \frac{1}{2} \sum_{i=0}^{3} \theta_i \sigma_i$ we obtain

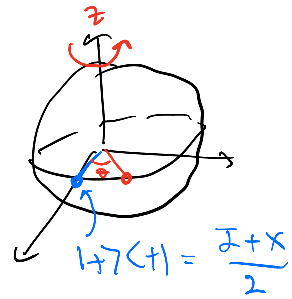
$$e^{-iA} = e^{-i/2 \sum_i \theta_i \sigma_i} = e^{-i\theta_0/2} \underbrace{e^{-i/2 \sum_{i=1}^{3} \theta_i \sigma_i}}$$

$$e^{-i\theta_0 \sigma_0 /2} = e^{-i\theta_0/2}\, I_{\mathbb{C}^2}$$

see Ex in "operator functions"

a) $R_z(\theta) = e^{-i\theta z/2}$ (i=3) action on Bloch sphere

$$R_z(\theta)\left(\frac{I+X}{2}\right)R_z(\theta)^\dagger = \frac{1}{2}\left(I+\cos\theta\, X+\sin\theta\, Y\right)$$



$$e^{-i\theta Z/2} \; X \; e^{i\theta Z/2} =$$

$$= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}}$$

$$= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} 0 & e^{-i\theta/2} \\ e^{i\theta/2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} = \cos\theta\, X + \sin\theta\, \underbrace{Y}_{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$e^{-i\theta Z/2} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

So any unitary can be written as

$$U = e^{i\theta'} \underbrace{e^{-i\theta\, n\cdot\sigma/2}}_{R_n(\theta)\ :\ \text{rotation about } n\text{-axis.}}$$

n : unit vector, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$

$(\theta_1, \theta_2, \theta_3) = \theta n$

<u>Aside</u> $U(\mathbb{C}^n)$ acts on $\mathrm{Herm}(\mathbb{C}^n)$ by conjugation:

$$A \mapsto U A U^\dagger .$$

$$P(\mathbb{C}^n) \subseteq \mathrm{Herm}(\mathbb{C}^n).$$

<u>Alternative form of Postulate 2</u>

Time evolution is described by the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle \qquad (*)$$

(We set Planck constant $\hbar = 1$)

To see this diagonalize H

$$H = \sum_i \lambda_i |i\rangle\langle i|$$

$$|\psi\rangle = \sum_i \psi_i |i\rangle$$

Then $(*)$ means $(\hbar = 1)$

$$\frac{d\psi_i}{dt} = -i\lambda_i \psi_i$$

i.e. $\psi_i(t) = e^{-i\lambda_i t}\psi_i(0)$

If $|\psi'\rangle = \sum_i \psi_i(t_2)\, |i\rangle$ then

$$|\psi'\rangle = \sum_i e^{-i\lambda_i t_2}\, \psi_i(0)\, |i\rangle$$

$$= \sum_i e^{-i\lambda_i t_2}\, e^{i\lambda_i t_1}\, \psi_i(t_1)\, |i\rangle$$

$$= \sum_i e^{-i\lambda_i (t_2 - t_1)}\, \psi_i(t_1)\, |i\rangle$$

$$= \underbrace{e^{-iH(t_2-t_1)}}_{\sum_i e^{-i\lambda_i (t_2-t_1)} |i\rangle\langle i|} |\psi\rangle$$

we obtain original formula of Postulate 2.

Jargon: H : Hamiltonian

$|i\rangle$: energy eigenstate

λ_i : energy

The eigenstate with the lowest energy is called the ground state.

Ex Consider

Pauli's

$$H = \boxed{\frac{1}{2} \sum_i \alpha_i \sigma_i} \qquad \alpha_i \in \mathbb{R}$$

$$\det H = \frac{1}{4}\left(\alpha_0^2 - |\alpha|^2\right) \qquad \alpha = (\alpha_1, \alpha_2, \alpha_3)$$

$$\mathrm{Tr}\, H = \alpha_0$$

Let λ_1 and λ_2 be eigenvalues of H.

$$\det H = \lambda_1 \lambda_2 \qquad \text{(see previous Ex)}$$
$$\operatorname{tr} H = \lambda_1 + \lambda_2$$

Then

$$\lambda_2 = \alpha_0 - \lambda_1$$

$$\lambda_1 (\alpha_0 - \lambda_1) = \frac{1}{4} \left(\alpha_0^2 - |\alpha|^2 \right)$$

When $\alpha_0 = 0$ we have

$$\lambda_2 = -\lambda_1$$

$$\lambda_1 = \frac{1}{2} |\alpha|$$

(In general
$$\lambda = \frac{1}{2} (\alpha_0 \pm |\alpha|)$$
)

So the eigenvals are $\pm |\alpha| / 2$.

Jargon:

In physics the spin of the electron is represented by

$$S_n = \sum_{i=1}^{3} n_i \sigma_i \quad \text{where} \quad \underbrace{|n|}_{\text{length of } n} = 1 \qquad [(n_1, n_2, n_3) \in \mathbb{R}^3]$$

and the magnetic field is represented

$$B = (B_1, B_2, B_3) \in \mathbb{R}^3$$

The Hermitian matrix

$$H = \gamma \, B \cdot S \qquad (\gamma > 0 \text{ constant})$$

$$= \gamma \sum_{i=1}^{3} B_i n_i \sigma_i$$

represents the Hamiltonian of an electron in a magnetic field

So $\alpha_i = 2\gamma B_i n_i$ and the ground state is

$$-|\alpha|/_2 = -\gamma \,|\underbrace{\sum_{i=1}^{3} B_i n_i}_{B\cdot n}|$$

(other eigenstate is $|\alpha|/_2$)

Postulate 3 - Measurements

Quantum measurements are described by a collection $\{M_m\}_m$ where $M_m \in L(V)$

$\uparrow$ finite #

such that

$$\sum_m M_m^{+} M_m = I_V .$$

The index m is interpreted as the outcome of the measurement

If the state of the system is $|\psi\rangle$ then

$$p(m) = \underbrace{\langle \psi | M_m^{+} M_m | \psi \rangle}_{Tr(M_m^{+} M_m |\psi\rangle\langle\psi|)} \quad \text{(Born rule)}$$

probability that m occurs

and the state of the system after

measurement

$$\frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^\dagger M_m |\psi\rangle}}.$$

Rem Recall that

$$\mathrm{Pos}(V) = \{ M^\dagger M \mid M \in L(V) \}$$

Therefore a quantum measurement can also be described as

$$\{ E_m \in \mathrm{Pos}(V) \}$$

s.t. $$\sum_m E_m = I_V .$$

These are known as positive operator-valued measure (POVM)

$$p(m) = \langle \psi | E_m | \psi \rangle .$$

Ex - Qubits $V = \mathbb{C}^2$

1) Any $E \in \mathrm{Pos}(\mathbb{C}^2)$ has the form

$$E = \frac{1}{2} \sum_{i=0}^{3} \alpha_i \sigma_i \quad \text{where} \quad \boxed{\alpha_0 \geq |\alpha|} \geq -\alpha_0$$

A POVM with two outcomes is given by

$$\boxed{E_0 = E \quad \text{and} \quad E_1 = I_V - E} \qquad E_0 + E_1 = I_V$$

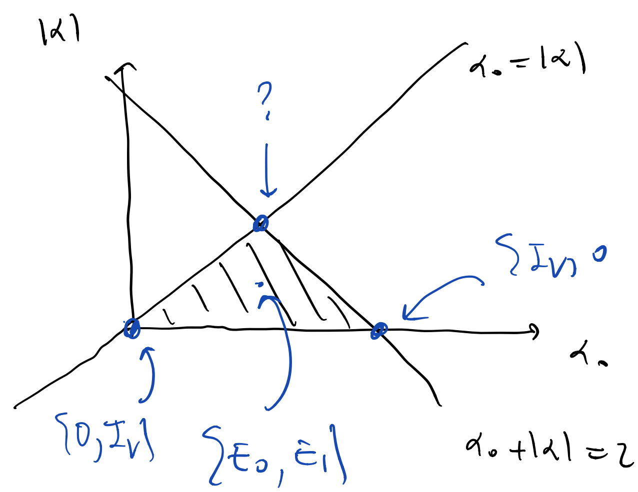
$$E_1 = \frac{1}{2}\sum_{i=0}^{3} p_i \sigma_i \qquad p_0 = 2-\alpha_0$$

$$p_i = -\alpha_i \qquad i = 1,2,3$$

s.t. $\quad p_0 \geq |p| \geq -p_0$

i.e. $\quad \boxed{2-\alpha_0 \geq |\alpha|} \geq -(2-\alpha_0)$

Picture



? : $\{E_0, E_1\}$ when $\boxed{\alpha_0 = 1 \;\&\; |\alpha| = 1}$

a) Measurement in computational basis

$$\{ |0\rangle\langle 0| \,,\, |1\rangle\langle 1| \} \qquad (i=3)$$

also known as Z-measurement

Specialized comment

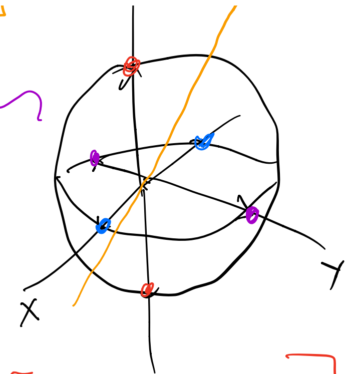
Hopf bundle $S^1 \to S^3 \to S^2$ $\qquad |\psi\rangle \sim \lambda|\psi\rangle$

Lie groups $U(1) \to \boxed{SU(2) \to SO(3)}$ $\qquad \lambda \in U(1)$

z

b) σ_i - meas. $\quad i = 1, 2, 3$ up | adjoint

Herm($\mathbb{C}^2$)



$$\left\{ \frac{I + \sigma_i}{2}, \frac{I - \sigma_i}{2} \right\}$$

$i = 1 \quad \{ |+\rangle\langle+|, |-\rangle\langle-| \}$

$i = 2 \quad \{ |i\rangle\langle i|, |-i\rangle\langle -i| \}$

$(|\pm i\rangle = |0\rangle \pm i|1\rangle)$

$$\boxed{P(\mathbb{C}^2) = S(\mathbb{C}^2)/U(1) = S^2}$$

2) Given $|\psi\rangle \in P(\mathbb{C}^2) = S^2$ Bloch sphere

$$|\psi\rangle\langle\psi| = \sum_{i=0}^{3} a_i \sigma_i \qquad a_0 = 1, \quad |\alpha| = 1$$

$$p(0) = \langle\psi| E_0 |\psi\rangle = \frac{1}{2}\left(\alpha_0 + \sum_{i=1}^{3} a_i \alpha_i\right)$$

$$p(1) = \langle\psi| E_1 |\psi\rangle = \frac{1}{2}\left(\beta_0 + \sum_{i=1}^{3} a_i \beta_i\right)$$

$\sum_{i=0}^{3} \beta_i a_i \qquad a_0 = 1$

a) Z-measurement

$|\psi\rangle = a|0\rangle + b|1\rangle$

$|a|^2 + |b|^2 = 1$

$$p(0) = \langle\psi|0\rangle\langle 0|\psi\rangle = |a|^2$$

$$p(1) = \langle\psi|1\rangle\langle 1|\psi\rangle = |b|^2$$

The state of the system after Z-measurement if the outcome is 0.

$$|\psi\rangle \xmapsto{|0\rangle\langle 0|} \frac{a}{|a|}|0\rangle \sim |0\rangle$$

$$|\psi\rangle \xmapsto{|1\rangle\langle 1|} \frac{b}{|b|}|1\rangle \sim |1\rangle$$

b) X - measurement $\quad |\psi\rangle = a'|+\rangle + b'|-\rangle$

$$p(0) = \langle\psi|+\rangle\langle+|\psi\rangle = |a'|^2$$

$$p(1) = \langle\psi|-\rangle\langle-|\psi\rangle = |b'|^2$$

$$|\psi\rangle \xmapsto{|+\rangle\langle+|} \frac{a'}{|a'|}|+\rangle$$

$$|\psi\rangle \xmapsto{|-\rangle\langle-|} \frac{b'}{|b'|}|-\rangle$$

Rem $\quad \{E_i\} \quad \sum_i E_i = I$

$$\sum_i p(i) = \sum_i \langle\psi|E_i|\psi\rangle$$

$$= \langle\psi|\sum_i E_i|\psi\rangle$$

$$= \langle\psi| I |\psi\rangle = \langle\psi|\psi\rangle = 1$$

unit vector

Projective measurements

A projective measurement is described by a Hermitian operator (called an observable):

$$M = \sum_m m P_m \qquad \text{(spectral decomposition)}$$

$\{m\}$ interpreted as outcomes

$$p(m) = \langle \psi | P_m | \psi \rangle$$

$$|\psi\rangle \xmapsto{\text{outcome } m} \frac{P_m |\psi\rangle}{\sqrt{\langle \psi | P_m | \psi \rangle}}$$

Rem This is a special case where

$$M_m = P_m$$

Then satisfy orgonality relations delta $\delta_{mm'} = \begin{cases} 1 & m = m' \\ 0 & m \neq m' \end{cases}$

$$P_m P_{m'} = \delta_{mm'} P_m .$$

Note that $M_m^\dagger M_m = P_m P_m = P_m$.

Normal $\quad AA^\dagger = A^\dagger A$

|

Herm $\quad A^\dagger = A$

|

Pos $\quad A = M^\dagger M$

|

Proj $\quad A^2 = A$

Expectation and standard deviation

The expectation (or average value) of a measurement is defined by

$$E(M) := \sum_m m \, p(m)$$

$$= \sum_m m \langle \psi | P_m | \psi \rangle$$

$$= \langle \psi | \sum_m m P_m | \psi \rangle$$

$$= \langle \psi | M | \psi \rangle$$

$$= \langle M \rangle \quad \text{(short hand notation)}$$

Standard deviation

$$\Delta(M) := \sqrt{\langle M^2 \rangle - \langle M \rangle^2}$$

$$= \sqrt{\langle (M - \langle M \rangle)^2 \rangle}$$

Rem The observable M defines a random variable:

a) A finite probability space consists of

- Ω finite set
- $p : \Omega \to \mathbb{R}_{\geq 0}$ such that

$$\sum_{\omega \in \Omega} p(\omega) = 1$$

b) A random variable is a function

$$X : \Omega \longrightarrow \mathbb{R}$$

$$p(x) = \underbrace{\sum_{\substack{\omega \in \Omega \\ \text{s.t.} \\ X(\omega) = x}}}_{p(X = x)} p(\omega) \qquad x \in \mathrm{im}(X)$$

$X = x$ means $\{\omega \in \Omega \mid X(\omega) = x\}$

c) Expectation

$$E(X) = \sum_{x \in X} x \, p(x)$$

Variance

$$\mathrm{var}(X) = E\left((X - E(X))^2 \right)$$

Standard deviation

$$\Delta(X) = \sqrt{\mathrm{var}(X)}$$

Construction

Let M be an observable and a state $|\psi\rangle$.

$$\Omega_M = \{ m \in \mathbb{R} \mid m \text{ is an eigenvalue of } M \}$$

$$P_\psi : \Omega \longrightarrow \mathbb{R}_{\geq 0}$$

$$P_\psi(m) = \langle \psi | M | \psi \rangle \qquad \text{Born rule}$$

$$X : \Omega \hookrightarrow \mathbb{R}$$
$$m \longmapsto m$$

Heisenberg principle

Let C and D be Hermitian operators.
$\quad x \qquad\qquad z$

Then

$$\Delta(C)\,\Delta(D) \geq \frac{|\langle \psi | [C,D] | \psi \rangle|}{2}.$$

Proof Set $A = C - \langle C \rangle I_V$ and $B = D - \langle D \rangle I_V$

Note that A and B are still Hermitian

$$[A,B] = [C - \langle C \rangle, D - \langle D \rangle] = [C,D]$$

$$\langle A^2 \rangle = \langle (C - \langle C \rangle)^2 \rangle = \Delta(C)^2$$

$$\langle B^2 \rangle = \langle (D - \langle D \rangle)^2 \rangle = \Delta(D)^2$$

We want

$$\langle A^2\rangle\langle B^2\rangle \geq \frac{|\langle\psi|[A,B]|\psi\rangle|^2}{4}$$

a) $\langle A^2\rangle\langle B^2\rangle \geq |\langle\psi|AB|\psi\rangle|^2$

by Cauchy - Schwarz inequality applied

to $|v\rangle = A|\psi\rangle$ and $|w\rangle = B|\psi\rangle$

Recall $\quad \|v\|\,\|w\| \geq \langle v|w\rangle$

b) Note $\quad \langle AB\rangle = \langle BA\rangle^*$

$$\langle [A,B]\rangle = \langle AB\rangle - \langle AB\rangle^*$$

$$\langle \{A,B\}\rangle = \langle AB\rangle + \langle AB\rangle^*$$

$$\underbrace{|\langle [A,B]\rangle|^2}_{(\langle AB\rangle - \langle AB\rangle^*)(\langle AB\rangle^* - \langle AB\rangle)} + \underbrace{|\langle \{A,B\}\rangle|^2} = 4\,|\langle AB\rangle|^2$$

Combining (a) & (b)

$$\langle A^2\rangle\langle B^2\rangle \geq |\langle AB\rangle|^2 \geq \frac{1}{4}|\langle [A,B]\rangle|^2$$

∎

Postulate 4 – composite systems

The Hilbert space of a composite system is the tensor product of the Hilbert spaces of the components:

$$V_{\text{composite}} = V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

composite system consisting of n-components

Most of the time we will consider systems with two components (bipartite).

A state $|\psi\rangle \in V_1 \otimes V_2$ is called an entangled state if

$$|\psi\rangle \neq \underbrace{|\psi_1\rangle \otimes |\psi_2\rangle}_{\text{product state}}$$

Ex $\quad V = \mathbb{C}^2 \otimes \mathbb{C}^2 \quad$ 2-qubits

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad \text{is entangled.}$$

Suppose $|\beta_{00}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ where

$$|\psi_1\rangle = a_0 |0\rangle + a_1 |1\rangle$$

$$|\psi_2\rangle = b_0 |0\rangle + b_1 |1\rangle$$

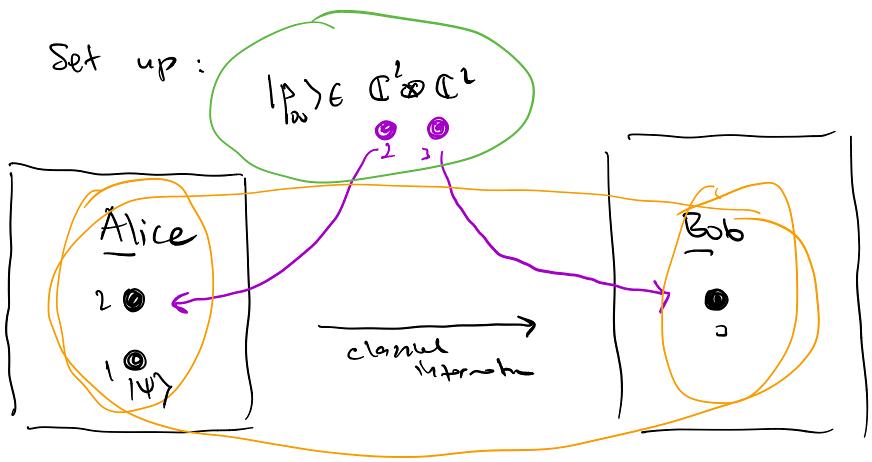
$$|\psi_1\rangle \otimes |\psi_2\rangle = \underbrace{a_0 b_0}_{1/\sqrt{2}}|00\rangle + \underbrace{a_0 b_1}_{0}|01\rangle + \underbrace{a_1 b_0}_{0}|10\rangle + \underbrace{a_1 b_1}_{1/\sqrt{2}}|11\rangle$$

$$\left.\begin{array}{llll} a_0 & \text{and} & b_0 & \neq 0 \\ a_0 & \text{or} & b_1 & = 0 \\ a_1 & \text{or} & b_0 & = 0 \\ a_1 & \text{and} & b_1 & \neq 0 \end{array}\right\} \quad \text{no solution!}$$

Applications

1) Quantum teleportation

Set up :



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Assumptions

1) Alice and Bob are physically separated

2) Alice does not know the state $|\psi\rangle$.

This means

$$\boxed{|\psi\rangle = \alpha|0\rangle + \beta|1\rangle}$$

Alice does not know α and β.

3) Alice can only send Bob classical information, i.e. sequence of 0 and 1's.

Question

Can Alice send $|\psi\rangle$ to Bob?

the information of the state.

Procedure

1) Consider

$$|\psi_0\rangle = |\psi\rangle |\beta_{00}\rangle \in \underbrace{\mathbb{C}^{2}_{(1)} \otimes \mathbb{C}^{2}_{(2)}}_{\text{Alice}} \otimes \underbrace{\mathbb{C}^{2}_{(3)}}_{\text{Bob}}$$

$$= (\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right)$$

2) Apply the unitary operators

a) CNOT gate to qubit (1) and (2)

$$C(X) : \underset{\text{control}}{\mathbb{C}^2} \otimes \underset{\text{target}}{\mathbb{C}^2} \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$\begin{array}{ccc} |00\rangle & & |00\rangle \\ |01\rangle & \longmapsto & |01\rangle \\ |10\rangle & & |11\rangle \\ |11\rangle & & |10\rangle \end{array}$$

b) Hadamard gate to qubit (1)

$$H: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

$$|0\rangle \mapsto |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$$
$$|1\rangle \mapsto |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$$

Apply (a) on $|\psi_0\rangle$:

$$\underbrace{C(X)_{12}}_{C(X)_{12} \otimes I_{\mathbb{C}^2}} |\psi_0\rangle = \frac{1}{\sqrt{2}}\Big(\alpha\big(|000\rangle + |011\rangle\big) + \beta\big(|110\rangle + |101\rangle\big)\Big)$$

Apply (b)

$$H_1\, C(X)_{12} |\psi_0\rangle = \frac{1}{\sqrt{2}}\Big(\alpha\big(|+\rangle|00\rangle + |+\rangle|11\rangle\big) + \beta\big(|-\rangle|10\rangle + |-\rangle|01\rangle\big)\Big)$$

$$= \frac{1}{2}\Big[\alpha\big(|0\rangle + |1\rangle\big)\big(|00\rangle + |11\rangle\big) + \beta\big(|0\rangle - |1\rangle\big)\big(|10\rangle + |01\rangle\big)\Big]$$

$$|\psi_1\rangle = \frac{1}{2}\Big[|00\rangle_A(\alpha|0\rangle + \beta|1\rangle)_B + |01\rangle_A(\alpha|1\rangle + \beta|0\rangle)_B + |10\rangle_A(\alpha|0\rangle - \beta|1\rangle)_B + |11\rangle_A(\alpha|1\rangle - \beta|0\rangle)_B\Big]$$

3) Alice performs Z measurements on qubits (1) and (2)

$a\ b$	$\lvert\psi_{00}\rangle$	$Z^a X^b$
00	$\boxed{\alpha\lvert 0\rangle + \beta\lvert 1\rangle}$	I
01	$\alpha\lvert 1\rangle + \beta\lvert 0\rangle$	X
10	$\alpha\lvert 0\rangle - \beta\lvert 1\rangle$	Z
11	$\alpha\lvert 1\rangle - \beta\lvert 0\rangle$	ZX

What is happening:

$$|\psi_1\rangle = H_1\, C(X)_{12}\, |\psi_0\rangle \in (\mathbb{C}^2)^{\otimes 3}$$

$$\{ P_{ab} = |ab\rangle\langle ab| \mid a,b \in \{0,1\} \}$$

$$|\psi_1\rangle \overset{P_{ab}}{\longmapsto} \frac{P_{ab}|\psi\rangle}{\sqrt{\langle\psi|P_{ab}|\psi\rangle}}$$

$$P_{ab}|\psi_1\rangle = \frac{1}{2}|ab\rangle|\psi_{bob}\rangle \sim |ab\rangle|\psi_{bob}\rangle$$

4) Bob applies unitary operator

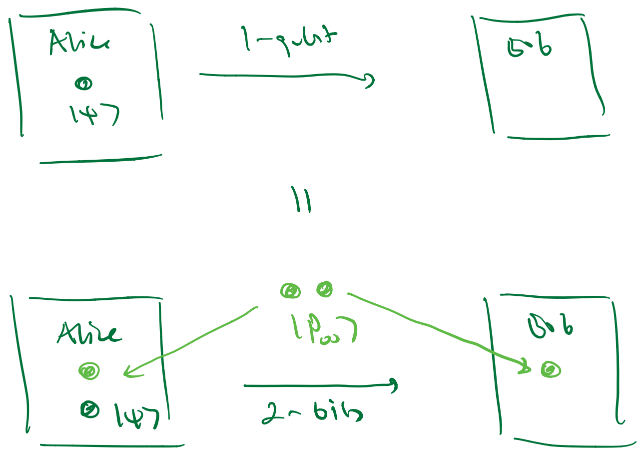
$$Z^a X^b |\psi_{Bob}\rangle = |\psi\rangle.$$

Conclusion

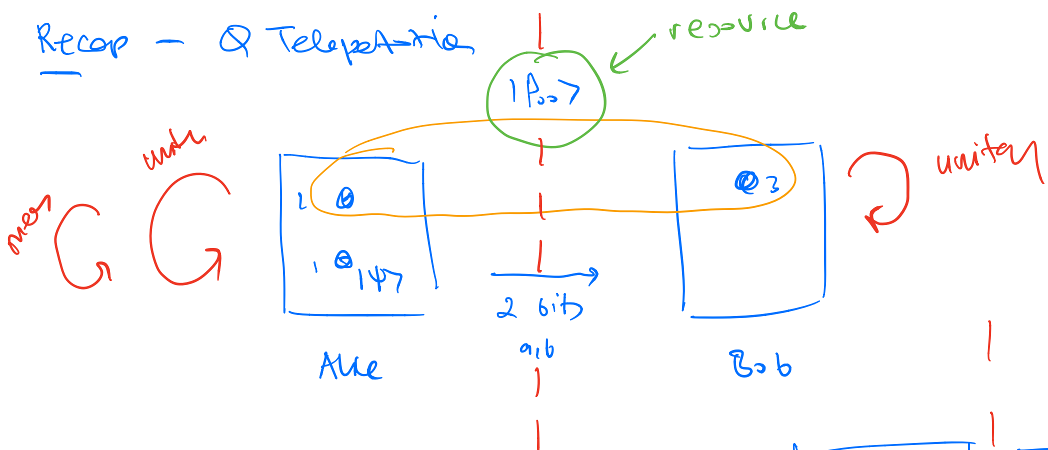
Alice applies $H_1\, C(X)_{12}$ to her qubits and measures them in Z-basis then sends the outcome $(a,b) \in \{0,1\}^2$. Bob applies $Z^a X^b$ to his qubit.

" $\Big(|\beta_{00}\rangle + (a,b) \Big) = |\psi\rangle$ "

entangled state — 2-bits of information — 1-qubit of quantum information i.e.



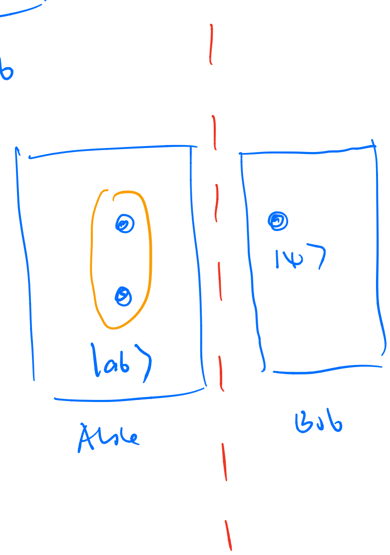
Recap – Q Teleportation



$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle$$

$$|\psi_1\rangle = H_1\, C(X)_{12}\, |\psi\rangle|\beta_{00}\rangle$$

$$= \sum_{a,b} |ab\rangle \underbrace{X^b Z^a |\psi\rangle}_{|\psi_{Bob}\rangle}$$



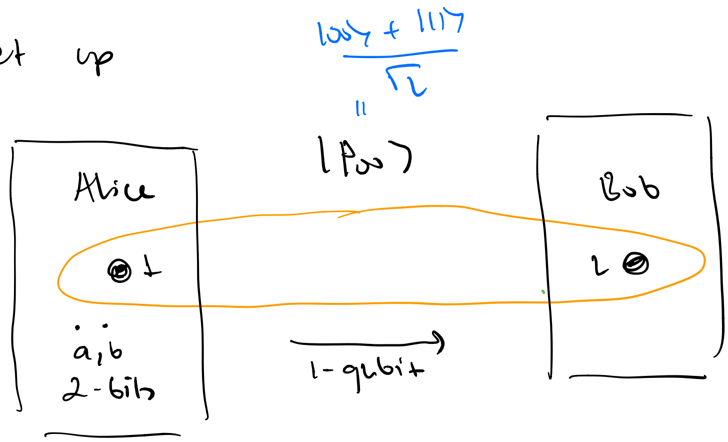
$$\{\, |ab\rangle\langle ab| \;\; | \;\; a,b \in \{0,1\} \,\}$$

$$|ab\rangle\langle ab|\psi_1\rangle = |ab\rangle \boxed{X^b Z^a |\psi\rangle}$$ apply $Z^a X^b$

$$Z^a X^b \left(X^b Z^a |\psi\rangle \right) = |\psi\rangle$$

2) Superdense coding

Set up



Assumption

1) Alice & Bob physically separated
2) Alice can send 1-qubit of quantum information to Bob.

Question: Can Alice send 2-bits of information to Bob?

Procedure

1) Alice applies unitary operator

a b	unitary
0 0	I
0 1	Z
1 0	X
1 1	ZX

$$\boxed{Z^b X^a}$$

$$\underbrace{Z^b X^a |\beta_{00}\rangle}_{|\beta_{ab}\rangle} = \underbrace{Z^b X^a}_{\text{applied to 1st qubit}} \left(\frac{\overset{\text{1st}\ \text{2nd}}{|00\rangle} + |11\rangle}{\sqrt{2}} \right)$$

$$= \frac{(-1)^{b \cdot a} |0+a\rangle |0\rangle + (-1)^{b(1+a)} |1+a\rangle |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \left\{ \begin{array}{lcc} & & ab \\ |00\rangle + |11\rangle & + \ + & 0\,0 \\ |00\rangle - |11\rangle & + \ - & 0\,1 \\ |10\rangle + |01\rangle & + \ + & 1\,0 \\ |01\rangle \underbrace{- |10\rangle} & + \ - & 1\,1 \end{array} \right.$$

$$|\beta_{ab}\rangle = (Z^b X^a)_1 \, |\beta_{00}\rangle$$

$\{ |\beta_{ab}\rangle \mid a, b \in \{0,1\} \}$ is called the Bell basis (orthonormal). verify this

2) Alice sends her qubit to Bob.

3) Bob measures in the Bell basis $\overbrace{\{ |\beta_{a'b'}\rangle \langle \beta_{a'b'}| \}}$

$$\underbrace{p(a'b')}_{\text{probability of obtaining outcome } a'b'} = \langle \beta_{ab} | \beta_{a'b'} \rangle \langle \beta_{a'b'} | \beta_{ab} \rangle$$

$$= \underbrace{\delta_{a,a'} \, \delta_{b,b'}}$$

" $\underbrace{|\beta_{00}\rangle}_{\text{resource}}$ + 1-qubit information = 2-bits of information "

3) Distinguishing quantum states

Set up

$$\{ |\psi_i\rangle \mid i \in I \}, \quad I = \{1, 2, \dots, n\}$$

Assumptions,

1) Alice and Bob know the states in the set.

Question

Alice chooses a state $|\psi_i\rangle$ from the set and sends this state to Bob.

Can Bob determine i from the state?

a) Suppose $\{|\psi_i\rangle\}$ orthonormal

Yes, Bob can measure $\{M_i\}_{i=0}^{n}$

$$\boxed{\begin{aligned} M_i &= |\psi_i\rangle\langle\psi_i| \\ M_0 &= I - \sum_{i=1}^{n} M_i \end{aligned}} \qquad i > 0$$

For $i' > 0$ $\quad \boxed{p(i') = \langle\psi_i | M_{i'} | \psi_i\rangle}$

$$= \delta_{ii'}$$

Bob can determine i with certainty.

2) General case : No.

Assume that there exists $\{M_j\}_{j\in J}$

and a function $f: J \to I$ such that

i) Bob measures $\{M_j\}$

$$\boxed{p(j) = \langle \psi_i | M_j^\dagger M_j | \psi_i \rangle}$$

ii) Bob guesses $i = f(j)$ with certainty.

More precisely Bob measures $\{E_i\}_{i\in I}$

$$\boxed{E_i = \sum_{\substack{j\in J \\ f(j)=i}} M_j^\dagger M_j}$$

and obtains

$$\boxed{p(i) = \langle \psi_i | E_i | \psi_i \rangle = 1 \quad \forall i \in I.}$$

We will see that this gives a contradiction:

If $|\psi_1\rangle$ & $|\psi_2\rangle$ are not orthogonal then

$$|\psi_2\rangle = \alpha |\psi_1\rangle + \beta |\varphi\rangle \qquad (*)$$

where $|\varphi\rangle$ is orthogonal to $|\psi_1\rangle$ and

$|\alpha|^2 + |\beta|^2 = 1$, $\boxed{|\beta| < 1}$.

By completeness $\sum_i E_i = I$ we have

$$1 = \langle \psi_1 | \psi_1 \rangle = \underbrace{\sum_i \langle \psi_1 | E_i | \psi_1 \rangle}$$

$\langle \psi_1 | E_i | \psi_1 \rangle = 0 \quad \forall\, i \neq 1$ $\quad i=1$

since $\langle \psi_1 | E_1 | \psi_1 \rangle = 1$.

In particular, $\underline{\sqrt{E_2}\, |\psi_1\rangle} = 0$ $\quad (\underline{\langle \psi_1 | E_2 | \psi_1 \rangle} = 0)$

Then

$$\langle \psi_2 | E_2 | \psi_2 \rangle \overset{(*)}{=} |\beta|^2 \langle \varphi | E_2 | \varphi \rangle$$

$$\leq |\beta|^2 \underbrace{\sum_i \langle \varphi | E_i | \varphi \rangle}_{\langle \varphi | \varphi \rangle = 1}$$

$$= |\beta|^2 < 1$$

This gives

$p(2) = \langle \psi_2 | E_2 | \psi_2 \rangle < 1$ contradiction

to the assumption that $p(2) = 1$.

Ex $\quad |\psi_1\rangle = |0\rangle \,, \quad |\psi_2\rangle = |+\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

$$\left.\begin{aligned} E_1 &= \frac{\sqrt{2}}{1+\sqrt{2}} \, |-\rangle\langle -| \qquad \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ E_2 &= \frac{\sqrt{2}}{1+\sqrt{2}} \, |1\rangle\langle 1| \\ E_3 &= I - E_1 - E_2 \end{aligned}\right\} \quad \{E_i\}$$

a) Bob is given $|\psi_1\rangle = |0\rangle$

$$\begin{aligned} p(1) &= \langle 0 | E_1 | 0 \rangle = 1/(1+\sqrt{2})\sqrt{2} \\ p(2) &= \langle 0 | E_2 | 0 \rangle = 0 \end{aligned}$$

$\rightarrow \quad p(3) = \langle 0 | E_3 | 0 \rangle = 1 - 1/(1+\sqrt{2})\sqrt{2}$

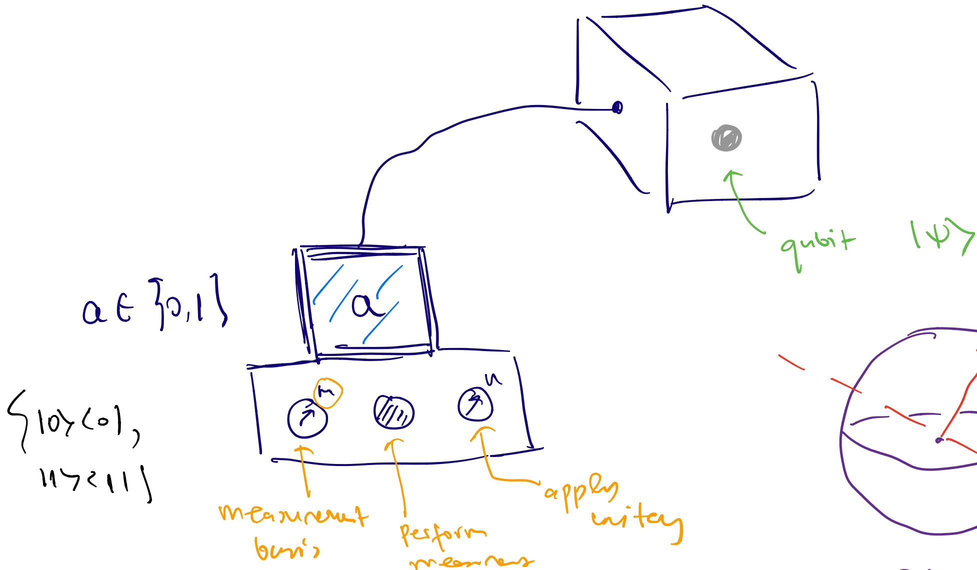
b) Bob is given $|\psi_2\rangle = |+\rangle$

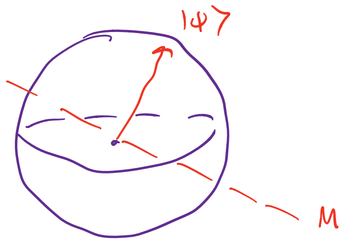
$$\begin{aligned} p(1) &= \langle + | E_1 | + \rangle = 0 \\ p(2) &= \langle + | E_2 | + \rangle = 1/(1+\sqrt{2})\sqrt{2} \end{aligned}$$

$\rightarrow \quad p(3) = \langle + | E_3 | + \rangle = 1 - 1/(1+\sqrt{2})\sqrt{2}$

Bob never misidentifies the state but sometimes he $\boxed{\text{can not infer anything.}}$

More on measurements





Bloch sphere

1) $|\psi\rangle = |0\rangle$

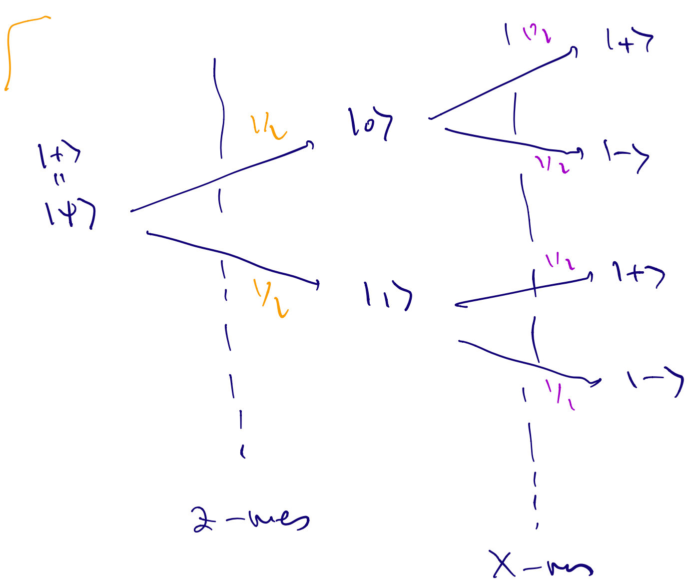
$p(0) = 1$

$p(1) = 0$

2) $|\psi\rangle = |+\rangle$

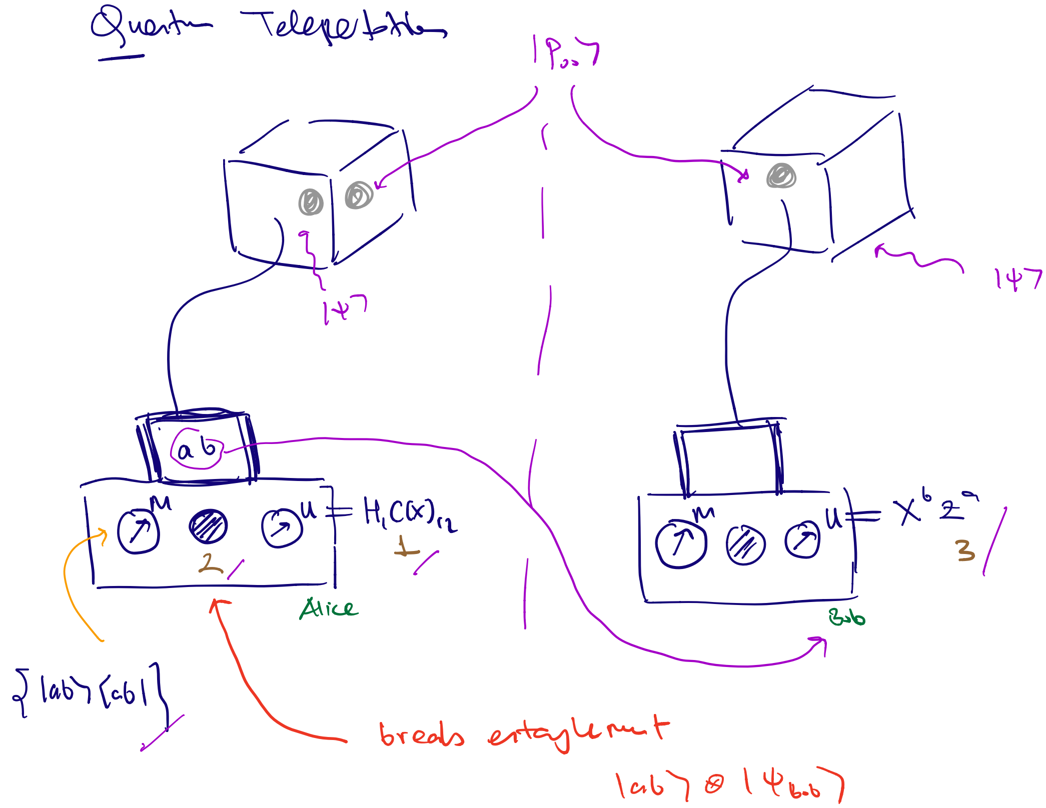
$p(0) = 1/2$

$p(1) = 1/2$

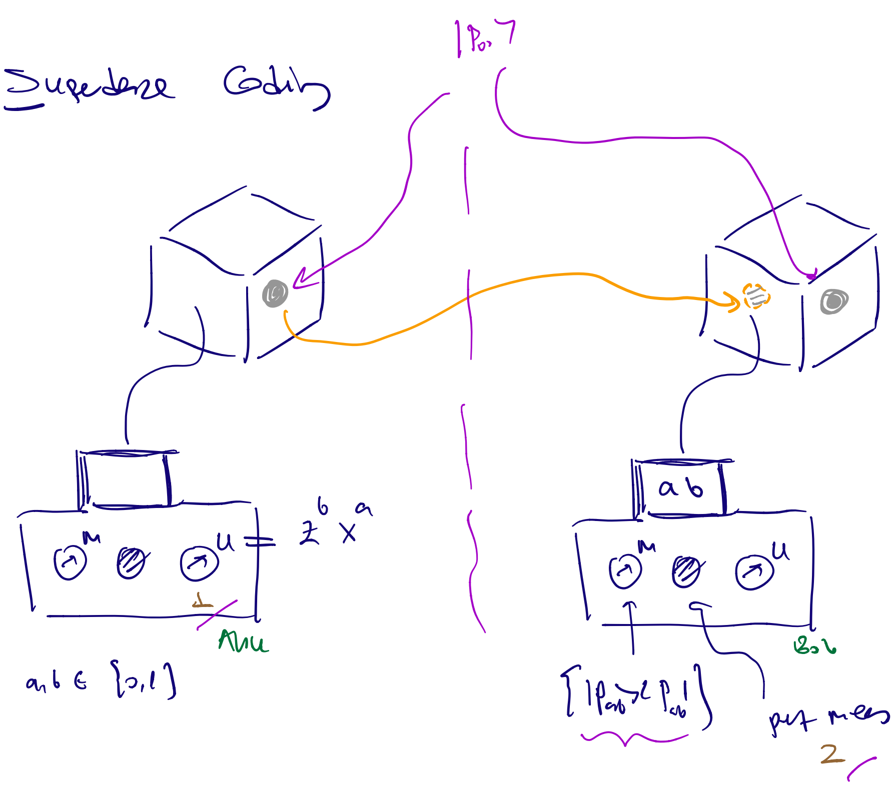


Z	X	
0	0	1/4
0	1	1/4
1	0	1/4
1	1	1/4

Quantum Teleportation



Superdense Coding



Density operators

1) A density operator is a linear operator $A \in L(V)$ such that

a) $A \in Pos(V)$

b) $Tr(A) = 1$.

Notation: $Den(V) \; \big(\subseteq Pos(V) \subseteq Herm(V) \big)$

2) Note that $Den(V)$ contains projectors, $A \in Proj(V)$, with $Tr(A) = 1$.

$(V_A : \dim V_A = 1.)$

Such a projector can be written as

$$\boxed{A = |\psi\rangle\langle\psi|} \quad \text{for some } |\psi\rangle \in V$$

More over for $\{ |\psi_i\rangle \}_i$ we will see that

$$\boxed{\sum_i p_i |\psi_i\rangle\langle\psi_i|} \in Den(V)$$

where $p_i \geq 0$ & $\sum_i p_i = 1$.

Rem a) These are not projectors in general.

b) Let $X \subseteq \mathbb{R}^n$ be a subset

$$conv(X) = \Big\{ \sum_{x \in X} p_x \, x \;\Big|\; p_x \geq 0 \;\&\; \sum_x p_x = 1 \;\&\; p_x \neq 0 \text{ for finitely many } x \in X \Big\}$$

is called the convex hull of X.

We will take $P(V) \subseteq \mathrm{Herm}(V)$ and consider

$$\mathrm{conv}(P(V)).$$

$\boxed{\text{Theorem}}$

$$\mathrm{Den}(V) = \mathrm{conv}(P(V))$$

Proof ($\Leftarrow$)

Given $\sum_i p_i |\psi_i\rangle\langle\psi_i|$

1) $\langle\varphi| \left(\sum_i p_i |\psi_i\rangle\langle\psi_i| \right) |\varphi\rangle$

$$= \sum_i p_i \underbrace{\langle\varphi|\psi_i\rangle\langle\psi_i|\varphi\rangle}_{|\langle\varphi|\psi_i\rangle|^2} \geq 0$$

2) $\mathrm{Tr}\left(\sum_i p_i |\psi_i\rangle\langle\psi_i| \right) = \sum_i p_i \,\mathrm{Tr}(|\psi_i\rangle\langle\psi_i|)$

$$= \sum_i p_i \underbrace{\langle\psi_i|\psi_i\rangle}_{1}$$

$$= \sum_i p_i = 1$$

($\Rightarrow$) Let $\rho \in \mathrm{Den}(V)$. Since ρ is normal spectral decomposition gives

$$\rho = \sum_j \lambda_j |j\rangle\langle j| \qquad \lambda_j \in \mathbb{R}_{\geq 0}$$

Since $\mathrm{Tr}(\rho) = 1$ we have

$$\mathrm{Tr}\left(\sum_j \lambda_j |j\rangle\langle j| \right) = \sum_j \lambda_j = 1$$

Therefore

$$\rho = \sum_j \lambda_j \underbrace{|j\rangle\langle j|} \in \mathrm{Conv}(P(V)).$$

$\boxed{\checkmark}$

Revisiting the postulates

$$\boxed{\text{P1}: \quad \text{States are given by } \rho \in \mathrm{Den}(V).}$$

Ensembles of states

a) The collection $\{ p_i , |\psi_i\rangle \}_i$ where

$p_i \geq 0$, $\sum_i p_i = 1$ is called an ensemble of pure states:

The system is in one of the states $|\psi_i\rangle$ with probability p_i.

Jargon $|\psi\rangle$: pure state

ρ : mixed state.

b) $\mathrm{Den}(V) = \mathrm{Conv}(P(V))$ is convex set

$$\rho = \sum_i p_i \rho_i \in \mathrm{Den}(V)$$

If ρ_i corresponds to $\{ p_j^{(i)}, |\psi_j^{(i)}\rangle \}_j$

then ρ " $\{ p_i p_j^{(i)}, |\psi_j^{(i)}\rangle \}_{i,j}$

P2 Unitary evolution

$$\rho \mapsto U \rho U^\dagger \qquad \left(\begin{array}{l} |\psi\rangle \mapsto U|\psi\rangle \\ |\psi\rangle\langle\psi| \mapsto U|\psi\rangle\langle\psi|U^\dagger \end{array} \right)$$

P3 Measurements $\{M_m\}_m$ $\sum_m M_m = I_V$

$$p(m) = \mathrm{Tr}(M_m^\dagger M_m \rho)$$

$$\rho \mapsto \frac{M_m \rho M_m^\dagger}{p(m)}$$

$$|\psi\rangle \mapsto \boxed{\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}}$$

$$|\psi\rangle\langle\psi| \mapsto \frac{M_m|\psi\rangle\langle\psi|M_m^\dagger}{\boxed{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \quad \underbrace{\mathrm{Tr}(M_m^\dagger M_m |\psi\rangle\langle\psi|)}_{p(m)}$$

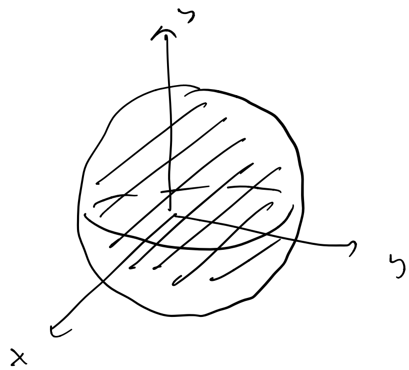
P4 Composite system

$$V = V_1 \otimes \cdots \otimes V_n$$

$$\rho \in \mathrm{Den}(V)$$

Ex $V = \mathbb{C}^2$

$$\mathrm{Den}(\mathbb{C}^2) = \left\{ \frac{1}{2}\left(\sigma_0 + \sum_{i=1}^{3} \alpha_i \sigma_i \right) \;\middle|\; |\alpha| \leq 1 \right\}$$



3-ball whose boundary is the Bloch sphere

$$\mathrm{Den}(\mathbb{C}^2) = \mathrm{Conv}(P(\mathbb{C}^2)) = \mathrm{Conv}(S^2)$$

Characterization of pure states

1) $\mathrm{Tr}(\rho^2) \leq 1 \qquad \forall \rho \in \mathrm{Den}(V)$

$\boxed{2)}$ $\mathrm{Tr}(\rho^2) = 1 \iff \rho$ is a pure state (i.e. $\rho^2 = \rho$.)

Proof 1)

$$\rho = \left(\sum_j \lambda_j |j\rangle\langle j| \right)$$

$$\rho^2 = \sum_j \lambda_j^2 |j\rangle\langle j|$$

$$\mathrm{Tr}(\rho^2) = \sum_j \lambda_j^2 \leq \sum_j \lambda_j = 1$$

2) Restatement:

$$\boxed{\sum_j \lambda_j^2 = 1 \iff \lambda_j \in \{0,1\} \quad \forall j}$$

$$\sum_j \lambda_j = 1$$

$\Leftarrow$ Easy.

$\boxed{\Rightarrow}$ If $0 < \lambda_{j'} < 1$ for some j' then

$$\sum_j \lambda_j^2 < \sum_j \lambda_j = 1$$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2} + \frac{1}{2} = 1$

$\lambda \ , \ \lambda'$ $\lambda + \lambda' = 1$

$\lambda^2 + (\lambda')^2$

$\frac{1}{4} + \frac{1}{4} < 1$

Ex - Qubit

Let $\rho = \frac{1}{2}\left(\sigma_0 + \sum_{i=1}^{3} \alpha_i \sigma_i \right)$ $\quad |\alpha| \leq 1$

$$\mathrm{Tr}(\rho^2) = \frac{1}{4} \mathrm{Tr}\left(\left(1 + \sum_{i=1}^{3} \alpha_i^2 \right) \sigma_0 \right)$$

$$= \frac{1 + |\alpha|^2}{2}$$

ρ is pure $\iff \frac{1}{2}(1 + |\alpha|^2) = 1$, i.e. $|\alpha| = 1$.

Unitary freedom in the ensemble

Let $\rho \in \mathrm{Den}(V)$

$$\rho = \sum_{i=1}^{N} |\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i| = \sum_{i=1}^{N} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$$

$$\Longleftrightarrow \quad \boxed{|\tilde{\psi}_i\rangle = \sum_j U_{ij} |\tilde{\varphi}_j\rangle}$$

where U_{ij} : entries in a unitary matrix U.

Rem We will apply this to ensembles

$$\{p_i, |\psi_i\rangle\} \quad \text{and} \quad \{q_i, |\varphi_i\rangle\}$$

by setting
$$|\tilde{\psi}_i\rangle = \sqrt{p_i}\,|\psi_i\rangle$$
$$|\tilde{\varphi}_i\rangle = \sqrt{q_i}\,|\varphi_i\rangle$$

The ensembles correspond to ρ $\Longleftrightarrow$

$$\sqrt{p_i}\,|\psi_i\rangle = \sum_j U_{ij} \sqrt{q_j}\,|\varphi_j\rangle .$$

Proof $(\Leftarrow)$

Suppose $|\tilde{\psi}_i\rangle = \sum_j U_{ij} |\tilde{\varphi}_j\rangle$

$$\underbrace{\boxed{\sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|}}_{(1)} = \sum_i \Big(\sum_j U_{ij} |\tilde{\varphi}_j\rangle\Big)\Big(\sum_k U_{ik}^* \langle\tilde{\varphi}_k|\Big)$$

$VW^\dagger$

V

$$= \sum_{i,j,k} U_{ij} \, U_{ik}^{*} \, |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_k|$$

$$= \sum_{j,k} \underbrace{\left(\sum_i (U^{\dagger})_{ki} \, U_{ij} \right)}_{\delta_{kj}} |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_k|$$

$$= \boxed{\sum_j |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_j|} \quad (2) \qquad \xrightarrow{W} \qquad \boxed{\sum_k |\tilde{k}\rangle\langle\tilde{k}|} \quad (3)$$

($\Rightarrow$) Suppose $\rho = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_j |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_j|$

Let $\rho = \sum_{k=1}^{M} \lambda_k |k\rangle\langle k|$ be the spectral decomposition. Let $|\tilde{k}\rangle = \sqrt{\lambda_k}\, |k\rangle$. $(\lambda_k > 0)$

Claim $|\tilde{\psi}_i\rangle, |\tilde{\varphi}_i\rangle \in \mathrm{Span}\{ |\tilde{k}\rangle \}_k$

This follows from the observation that if $|\psi\rangle$ satisfies $\langle\psi|\tilde{k}\rangle = 0 \ \forall k$ then

$$0 = \langle\psi|\rho|\psi\rangle = \sum_i \underbrace{\langle\psi|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|\psi\rangle}_{|\langle\psi|\tilde{\psi}_i\rangle|^2}$$

$$\underbrace{|\langle\psi|\tilde{\psi}_i\rangle|^2}_{0} \quad \forall i.$$

Therefore $\boxed{|\tilde{\varphi}_i\rangle = \sum_k c_{ik} |\tilde{k}\rangle}$

We have

$$\boxed{\sum_k |\tilde{k}\rangle\langle\tilde{k}|} = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$$

$$= \sum_i \left(\sum_k c_{ik} |\tilde{k}\rangle\right)\left(\sum_\ell c_{i\ell}^* \langle\tilde{\ell}|\right)$$

$$= \sum_{k,\ell} \underbrace{\boxed{\left(\sum_i c_{ik} c_{i\ell}^*\right)}} \boxed{|\tilde{k}\rangle\langle\tilde{\ell}|}$$

The operators $\{|\tilde{k}\rangle\langle\tilde{\ell}|\}_{k,\ell}$ are linearly independent in $\boxed{L(V)}$. In fact, they are orthogonal

$$(|\tilde{k}\rangle\langle\tilde{\ell}|, |\tilde{k}'\rangle\langle\tilde{\ell}'|) = \lambda_k \lambda_\ell \delta_{\tilde{k}\tilde{k}'} \delta_{\tilde{\ell}'\tilde{\ell}}$$

Therefore $\boxed{\sum_i c_{ik} c_{i\ell}^* = \delta_{k\ell}}$.

$$V = \left[\begin{array}{ccc|ccc} c_{11} & \cdots & c_{1M} & & & \\ \vdots & & \vdots & [\;] & \cdots & [\;] \\ c_{N1} & \cdots & c_{NM} & & & \end{array}\right] \quad N\times N \text{ matrix}$$

C (the first M columns), $N-M$ (the remaining columns)

Note that

$$N \geq \dim \operatorname{Im}\left(\sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|\right)$$

$$= \dim \operatorname{Im}\left(\sum_k |\tilde{k}\rangle\langle\tilde{k}|\right) = M$$

We have

$$\boxed{|\tilde{u}_i\rangle = \sum_{k=1}^{N} V_{ik} |\tilde{k}\rangle}$$

$$\left(|\tilde{k}\rangle = 0 \quad \text{for } k \geq m+1 \right)$$

Similarly

$$|\tilde{\varphi}_j\rangle = \sum_k W_{jk} |\tilde{k}\rangle$$

for some unitary W.

$$\hookrightarrow |\tilde{k}\rangle = \sum_j (W^\dagger)_{kj} |\varphi_j\rangle$$

Therefore

$$|\tilde{\varphi}_i\rangle = \sum_k V_{ik} |\tilde{k}\rangle$$

$$= \sum_{k,j} V_{ik} (W^\dagger)_{kj} |\tilde{u}_j\rangle$$

$$= \sum_j U_{ij} |\tilde{\varphi}_j\rangle$$

where $U = V W^\dagger$. ☒

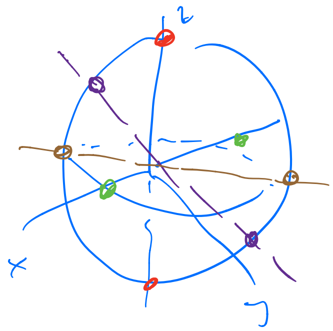
Rem $A, B \in U(L)$ $\quad A^\dagger = A^{-1}, \quad B^\dagger = B^{-1}$

$$(AB)^\dagger = B^\dagger A^\dagger = B^{-1} A^{-1} = (AB)^{-1}$$

$AB \in U(L)$.

Ex

$$\rho = I/2 \qquad \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$



$$\rho = \overbrace{\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|}$$

$$\rho = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$

$$\Big\{\Big(\frac{1}{2}, \overbrace{|0\rangle\langle 0|}^{|\tilde\psi_1\rangle\langle\tilde\psi_1|}\Big), \Big(\frac{1}{2}, \overbrace{|1\rangle\langle 1|}^{|\tilde\psi_2\rangle\langle\tilde\psi_2|}\Big)\Big\}$$ and

$$\Big\{\Big(\frac{1}{2}, \overbrace{|+\rangle\langle +|}^{|\tilde\varphi_1\rangle\langle\tilde\varphi_1|}\Big), \Big(\frac{1}{2}, \overbrace{|-\rangle\langle -|}^{|\tilde\varphi_2\rangle\langle\tilde\varphi_2|}\Big)\Big\}$$ represent the

same density operator $\rho = I/2$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$ usually denoted by H, called the Hadamard gate.

Partial trace

Partial trace is defined to be the unique linear map

$$\mathrm{Tr}_{V_1} : L(V_1 \otimes V_2) \longrightarrow L(V_2)$$

determined by

$$\mathrm{Tr}_{V_1}(A \otimes B) = \mathrm{Tr}(A)\, B$$

Similarly we can define Tr_{V_2}.

$$\mathrm{T}_{V_2} : L(V_1 \otimes V_2) \longrightarrow L(V_1)$$

Reduced density operator

Let $\rho \in \mathrm{Den}(V_1 \otimes V_2)$

Usually V_1 and V_2 describe physical systems.

The reduced density operator $\rho^{(1)} \in \mathrm{Den}(V_1)$ defined by

$$\rho^{(1)} = \mathrm{Tr}_{V_2}(\rho)$$

similarly

$$\rho^{(2)} = \mathrm{Tr}_{V_1}(\rho)$$

Ex $\quad \rho = |\beta_{00}\rangle\langle\beta_{00}|$

$$= \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \right)$$

$$\rho^{(1)} = \mathrm{Tr}_{V_2}(\rho)$$

$$= \frac{1}{2}\Big(|0\rangle\langle 0| \overbrace{\mathrm{Tr}(|0\rangle\langle 0|)}^{\langle 0|0\rangle} + |0\rangle\langle 1| \cancel{\mathrm{Tr}(|0\rangle\langle 1|)}$$

$$+ |1\rangle\langle 0| \cancel{\mathrm{Tr}(|1\rangle\langle 0|)} + |1\rangle\langle 1| \mathrm{Tr}(|1\rangle\langle 1|) \Big)$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2} I$$

$$\mathrm{Tr}_{V_2}(|ab\rangle\langle cd|) = |a\rangle\langle c| \underbrace{\mathrm{Tr}(|b\rangle\langle d|)}_{\langle d|b\rangle}$$

$$= \boxed{\langle d|b\rangle} |a\rangle\langle c| .$$

$$\rho^{(2)} = \mathrm{Tr}_{V_1}(\rho) = I/2$$

Jargon $\quad \frac{1}{\dim V} I_V \in \mathrm{Den}(V)$ is called the completely mixed state.

$$\mathrm{Tr}\left(\left(\frac{1}{\dim V} I_V\right)^2\right) = \frac{1}{\dim V} \underbrace{\mathrm{Tr}\left(\frac{1}{\dim V} I_V\right)}_{1} = \frac{1}{\dim V} < 1$$

Alternative characterization of partial trace

Tr_{V_2} is the unique linear map that satisfies

$$Tr\left((\underset{L(V_1)}{A} \otimes I_{V_2}) \underset{L(V_1 \otimes V_2)}{\tilde{B}} \right) = Tr\left(A \left(Tr_{V_2} \tilde{B} \right) \right)$$

Because

$$Tr_{V_2} \tilde{B} = \sum_i \overbrace{Tr\left(M_i^\dagger \, Tr_{V_2} \tilde{B} \right)}^{(M_i,\ Tr_{V_2} \tilde{B})} M_i$$

$$= \sum_i \boxed{Tr\left((M_i^\dagger \otimes I_{V_2}) \tilde{B} \right)} M_i$$

where $\{M_i\}_i$ orthonormal basis for $L(V)$.

Physical interpretation

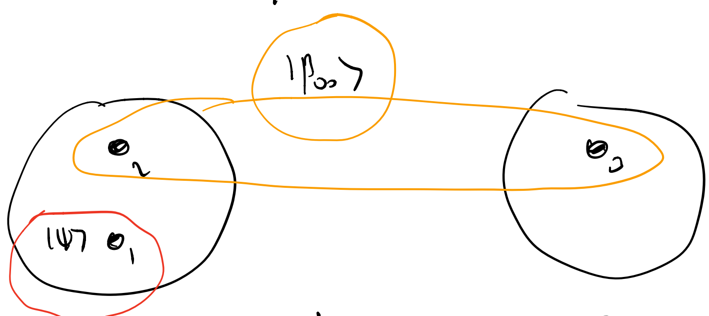
Let $\rho \in L(V_1 \otimes V_2)$ and $M \in Herm(V_1)$

$$M = \sum_\lambda \lambda \Pi_\lambda$$

$$Tr\left((\Pi_\lambda \otimes I_{V_2}) \rho \right) = Tr\left(\Pi_\lambda \rho^{(1)} \right)$$

That is measuring $M \otimes I_{V_2}$ on the composite system produces the same measurement statistics as measuring M on the first system.

Let us look at quantum teleportation:



Alice $V_1 = \mathbb{C}^2 \otimes \mathbb{C}^2$ Bob $V_2 = \mathbb{C}^2$

$$|\varphi_1\rangle = H_1\, C(X)_{12} \left(|\psi\rangle |\beta_{00}\rangle\right)$$

$$= \sum_{a,b} |ab\rangle\, X^b Z^a |\psi\rangle$$

After Alice measures in $\{|ab\rangle\langle ab|\}_{a,b}$

$$\underbrace{|ab\rangle\,|\beta_{ab}\rangle}_{(Z^b X^a)_{(1)} |\beta_{00}\rangle} \quad \text{with probability } 1/4$$

$a, b \in \{0,1\}$

This describes an ensemble $\left\{\left(\frac{1}{4},\ |ab\rangle|\beta_{ab}\rangle\right)\right\}_{a,b}$

$$\rho = \frac{1}{4} \sum_{a,b} \left(|ab\rangle\langle ab|\right)\left(|\beta_{ab}\rangle\langle \beta_{ab}|\right)$$

$$\rho^{(2)}_{Bob} = \mathrm{Tr}_{V_1}\, \rho = \boxed{I/2} \quad \text{(verify)}$$

This is the state of Bob's system after Alice has performed the measurement but before Bob has learned the measurement result.

The Schmidt decomposition

Theorem Let $|\psi\rangle \in V_1 \otimes V_2$ unit vector.

There exist orthonormal sets of vectors

- $\{ |i_A\rangle \in V_1 \}_i$
- $\{ |i_B\rangle \in V_2 \}_i$

such that

$$\boxed{|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle}$$

where $\lambda_i \in \mathbb{R}_{\geq 0}$ satisfy $\sum_i \lambda_i = 1$

Rem 1) $\{\lambda_i\}$ are called the Schmidt coefficients.

2) The reduced density operators are given by

$$\rho^{(1)} = \mathrm{Tr}_{V_2} \underbrace{|\psi\rangle\langle\psi|}_{\rho} = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$$

$$\rho^{(2)} = \mathrm{Tr}_{V_1} |\psi\rangle\langle\psi| = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$$

The eigenvalues of $\rho^{(1)}$ & $\rho^{(2)}$ are the same.

<u>Proof</u> Let $\{|j_A\rangle\}_{j=1}^{n}$ and $\{|k_B\rangle\}_{k=1}^{m}$ be orthonormal bases for V_1 & V_2.

We can expand

$$(*) \qquad |\psi\rangle = \sum_{j,k} A_{j,k}\, |j_A\rangle \otimes |k_B\rangle$$

Consider the $n \times m$ matrix A, and complete it to a square matrix by adding zeros:

$$\tilde{A} = \begin{cases} \begin{bmatrix} A_{11} & \cdots & A_{1m} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n1} & \cdots & A_{nm} & 0 & \cdots & 0 \end{bmatrix} & \text{if } m \leq n \\[2em] \begin{bmatrix} A_{11} & \cdots & \cdots & A_{1m} \\ \vdots & & & \vdots \\ A_{n1} & \cdots & \cdots & A_{nm} \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} & \text{if } m > n \end{cases}$$

Take the singular value decomposition

$$\tilde{A} = U D V \quad \text{where } D: \begin{cases} n \times n & n \le n \\ n \times m & m > n \end{cases}$$

Therefore

$$A_{jk} = \sum_i U_{ji} D_{ii} V_{ik}$$

Then we can substitute this in (*)

$$|\psi\rangle = \sum_{j,k,i} U_{ji} D_{ii} V_{ik} \; |j_A\rangle \otimes |k_B\rangle$$

Let us define

$$|i_A\rangle = \sum_j U_{ji} \; |j_A\rangle$$

$$|i_B\rangle = \sum_k V_{ik} \; |k_B\rangle$$

Set $\lambda_i = D_{ii}$.

$$\boxed{|\psi\rangle = \sum_i \lambda_i \; |i_A\rangle \otimes |i_B\rangle}$$

It remains to check that $\{|i_A\rangle\}_i$ and $\{|i_B\rangle\}_i$ are orthonormal :

$$\langle i'_A | i_A\rangle = \Big(\sum_{j'} \underbrace{U^*_{j'i'}}_{(U^\dagger)_{i'j'}} \langle j'_A|\Big)\Big(\sum_{j} U_{ji} |j_A\rangle\Big)$$

$$= \sum_j (U^\dagger)_{i'j}\, U_{ji} = \delta_{i'i}$$

$\langle i'_B | i_B\rangle = \delta_{i'i}$.

$\blacksquare$

<u>Jargon</u> The basis $\{|i_A\rangle\}_i$ and $\{|i_B\rangle\}_i$ are called the Schmidt basis for V_1 and V_2.

The number of non-zero λ_i is called the Schmidt number of $|\psi\rangle$.

<u>Rem</u> 1) $|\psi\rangle$ product state $\Leftrightarrow$ Schmidt # $= 1$.

$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ (product state)

then Schmidt # $= 1$. ($\lambda = 1$)

2) Schmidt # $= 1$ $\Leftrightarrow$ $\rho^{(1)}$ & $\rho^{(2)}$ pure.

$$\rho^{(1)} = \mathrm{Tr}_{V_2}\rho = \sum_i \lambda_i^2 \boxed{|i_A\rangle\langle i_A|}$$

$$\rho^{(2)} = \mathrm{Tr}_{V_1}\rho = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$$

Jargon: $|\psi\rangle$ is called maximally entangled if

$$\mathrm{Tr}_{V_1} |\psi\rangle\langle\psi| = \underbrace{I_{V_2} / \dim V_2}_{\text{completely mixed state}}$$

e.g. $$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Purification

Let $\boxed{\rho} \in \mathrm{Den}(V_1)$. Then there exists a Hilbert space V_2 and a pure state $|\psi\rangle \in V_1 \otimes V_2$ such that

$$\rho = \mathrm{Tr}_{V_2} |\psi\rangle\langle\psi|$$

The vector $|\psi\rangle$ is called a purification of ρ.

Proof: First diagonalize ρ:

$$\rho = \sum_i p_i |i\rangle\langle i|$$

Let V_2 be a vector space of $\dim V_2 = \dim V_1$. Choose an orthonormal basis $\{|e_i\rangle\}_i$.

Define $$|\psi\rangle = \sum_i \sqrt{p_i} \underbrace{|i\rangle |e_i\rangle}_{\text{Schmidt basis}}$$

Then

$$\mathrm{Tr}_{V_2}(|\psi\rangle\langle\psi|) = \sum_i p_i\, |i\rangle\langle i| = \rho$$

$\boxtimes$

<u>Unitary equivalence of purifications</u>

If $|\psi_1\rangle$ and $|\psi_2\rangle \in V_1 \otimes \boxed{V_2}$ satisfy

$$\mathrm{Tr}_{V_2} |\psi_1\rangle\langle\psi_1| = \mathrm{Tr}_{V_2} |\psi_2\rangle\langle\psi_2| = \rho$$

then there exists $u \in U(V_2)$ such that

$$|\psi_2\rangle = (I_{V_1} \otimes u)\, |\psi_1\rangle .$$

<u>Proof</u> Let $\{|a_i\rangle\}_i$ and $\{|b_i\rangle\}_i$ be orthonormal basis for V_1 and V_2.

Then

$$|\psi_1\rangle = \sum_{i,i'} \alpha_{ii'}\, |a_i\rangle |b_{i'}\rangle$$

$$|\psi_2\rangle = \sum_{i,i'} \beta_{ii'}\, |a_i\rangle |b_{i'}\rangle$$

Compute the partial traces

$$\mathrm{Tr}_{V_2} |\psi_1\rangle\langle\psi_1| = \sum \alpha_{ii}\, \alpha_{ii}^{*}\, |a_i\rangle\langle a_i|$$

$$\mathrm{Tr}_{V_2} |\psi_2\rangle\langle\psi_2| = \sum \beta_{ii}\, \beta_{ii}^{*}\, |a_i\rangle\langle a_i|$$

We will interpret partial traces differently:

Let

$$A_1 = \sum \alpha_{ii'} |a_i\rangle\langle b_{i'}|$$

$$A_2 = \sum \beta_{ii'} |a_i\rangle\langle b_{i'}|$$

Then

$$\mathrm{Tr}_{V_2} |\psi_1\rangle\langle\psi_1| = A_1 A_1^{\dagger}$$

$$\mathrm{Tr}_{V_2} |\psi_2\rangle\langle\psi_2| = A_2 A_2^{\dagger}$$

We have

$$A_1 A_1^{\dagger} = A_2 A_2^{\dagger} = \rho .$$

We will perform singular value decomposition:

a) First diagonalize ρ

$$\rho = \sum_i \lambda_i |v_i\rangle\langle v_i|$$

b) $$A_1 = \sqrt{\rho}\, U_1 = \sum_i \sqrt{\lambda_i} |v_i\rangle\langle v_i| U_1$$

$$A_2 = \sqrt{\rho}\, U_2 = \sum_i \sqrt{\lambda_i} |v_i\rangle\langle v_i| U_2$$

Let us define

$$\boxed{Y = U_1^{\dagger} U_2}$$

Then $\quad A_1 V = (\sqrt{e}\, u_1)(u_1^{+} u_2)$

$$= \sqrt{e}\, u_2 = A_2$$

Recall HW1 - Q3

$$\phi : L(\mathbb{C}^n, \mathbb{C}^n) \longrightarrow \mathbb{C}^n \otimes \mathbb{C}^n$$

$$|e_i\rangle\langle e_j| \longmapsto |e_i\rangle \otimes |e_j\rangle$$

$$(A \otimes B)\, \phi(C) = \phi(A C B^T)$$

We let $\boxed{U = V^T}$ and then

$$(I_{V_1} \otimes U)\,|\psi_1\rangle = (I_{V_1} \otimes V^T)\,|\psi_1\rangle$$

$$= |\psi_2\rangle$$

Apply HW-Q3 with $A = I_{V_1}$, $B = V^T$
$C = A_1$.

$$(I_{V_1} \otimes V^T)\,|\psi_1\rangle = \phi(\underbrace{I_{V_1} A_1 V}_{A_2}) = |\psi_2\rangle$$

$\blacksquare$

Bell inequalities

No local hidden-variable theory can produce all of the statistical predictions of quantum theory.

Random variables for commuting observables

Let $A, B \in \mathrm{Herm}(V)$ such that $AB = BA$.
By simultaneous diagonalization there exists a set of orthogonal projectors $\{ \Pi_{ij}^{AB} \}_{i,j}$

such that

$$\Pi_i^A = \sum_j \Pi_{ij}^{AB} \qquad \text{proj. onto } V_{\lambda_i^A}$$

$$\Pi_j^B = \sum_i \Pi_{ij}^{AB} \qquad \text{proj onto } V_{\lambda_j^B}$$

Let $\rho \in \mathrm{Den}(V)$.

Construct a probability space

$$\Omega = \{ (\lambda_i^A, \lambda_j^B) \mid i,j \}$$

$$p : \Omega \to \mathbb{R}_{\geq 0}$$

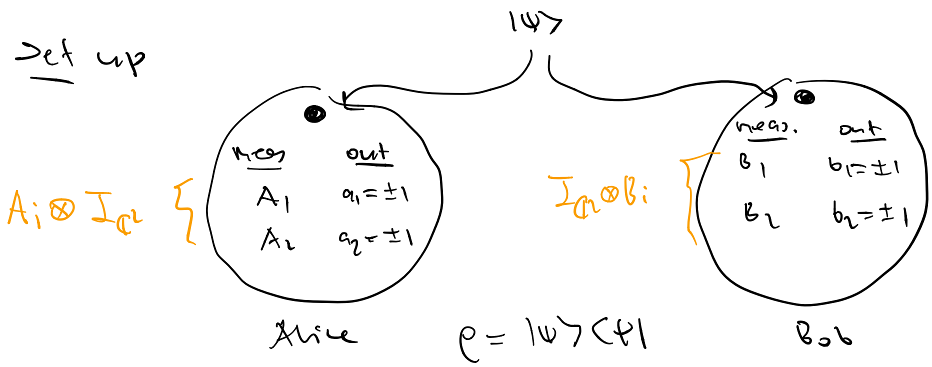
$$p(\lambda_i^A, \lambda_j^B) = \mathrm{Tr}(\rho \, \Pi_{ij}^{AB})$$

Define a random variable

$$\overline{AB} : \Omega \to \mathbb{R} \qquad (\lambda_i^A, \lambda_j^B) \mapsto \lambda_i^A \lambda_j^B .$$

CHSH inequality (Bell-type inequality)

Set up



Assumptions

1) Alice & Bob are physically separated.
2) Alice chooses to meas. A_1 or A_2 and Bob " B_1 or B_2.
3) They measure simultaneously.

Consider the random variables

$$\boxed{\overline{A_i B_j} : \{\pm 1\}^2 \longrightarrow \mathbb{R}, \quad i, j = 1, 2}$$

$$\boxed{P_\rho^{A_i B_j} : \{\pm 1\}^2 \longrightarrow \mathbb{R}_{\geq 0}}$$

Rem Quantum mechanically we cannot simultaneously measure non-com. observables.

So if $[A_1, A_2] \neq 0$ we can not define $\overline{A_1 A_2}$.

Goal: a) Assume that there is a local hidden variable model (HVM):

HVM

$$\Omega_{HVM} = \underbrace{\{\pm 1\}}_{a_1} \times \underbrace{\{\pm 1\}}_{a_2} \times \underbrace{\{\pm 1\}}_{b_1} \times \underbrace{\{\pm 1\}}_{b_2}$$

$$P_{HVM} : \Omega_{HVM} \longrightarrow \mathbb{R}_{\geq 0} \quad \text{(joint prob dist)}$$

We are assuming each observable takes definite outcomes specified by

$$(A_1, A_2, B_1, B_2) \mapsto (a_1, a_2, b_1, b_2) \quad \text{(realism)}$$

When we restrict to (A_1, B_1)

$$P_e^{A_1B_1}(a_1,b_1) = P_{HVM}^{A_1B_1}(a_1, b_1) = \sum_{a_2, b_2} P_{HVM}(a_1, a_2, b_1, b_2)$$

Similarly for all i, j

$$\boxed{P_{HVM}^{A_iB_j}(a_i, b_j) = P_e^{A_iB_j}(a_i, b_j).} \quad (*)$$

Alice's meas. outcome has no influence on Bob's meas out, and vice versa. (Locality)

In effect, we are assuming that each observable has a preassigned outcome which merely revealed by the act of meas.

b) Derive a contradiction by violating an inequality imposed by the Bell HVM.

Consider the random variables:

$$a_i b_j : \{\pm 1\}^2 \longrightarrow \mathbb{R}$$
$$(a_i, b_j) \longmapsto a_i b_j$$

$p^{A_i B_j}_{HVM}$ obtained p_{HVM} as in $(*)$.

Construct a new random variable

$$X(a_1, a_2, b_1, b_2) = a_1 b_1 + a_2 b_1 + a_1 b_2 - a_2 b_2$$

$$X : \{\pm 1\}^4 \longrightarrow \mathbb{R}$$

$$(\quad p_{HVM} : \{\pm 1\}^4 \longrightarrow \mathbb{R}_{\geq 0})$$

Derivation of the inequality

a) $\underbrace{a_1 b_1 + a_2 b_1 + a_1 b_2 - a_2 b_2}_{} = \pm 2$

$$(a_1 + a_2) b_1 + (a_1 - a_2) b_2$$

observe that

$$a_1 + a_2 = \begin{cases} 0 & \text{if } a_1 - a_2 = \pm 2 \quad \text{case I} \\ \pm 2 & \text{if } a_1 - a_2 = 0 \quad \text{case II} \end{cases}$$

$$= \begin{cases} \pm 2 \; b_2 & \text{case I} \\ \pm 2 \; b_1 & \text{case II} \end{cases} = \pm 2$$

b) $$E(X) = \sum_{\substack{a_1,a_2,\\ b_1,b_2}} P_{HVM}(a_1,a_2,b_1,b_2) \overbrace{X(a_1,a_2,b_1,b_2)}^{\pm 2}$$

$$\leq 2 \underbrace{\sum_{\substack{a_1,a_2\\ b_1,b_2}} P_{HVM}(a_1,a_2,b_1,b_2)}_{1} = 2$$

Note that the expectation is linear

$$E(X) = E(a_1 b_1) + E(a_2 b_1) + E(a_1 b_2) - E(a_2 b_2) \leq 2$$

c) Note that by $(*)$

$$E(a_i b_j) = E(\overline{A_i B_j}) \qquad \forall \; i,j.$$

Therefore

$$\boxed{E(X) = E(\overline{A_1 B_1}) + E(\overline{A_2 B_1}) + E(\overline{A_1 B_2}) - E(\overline{A_2 B_2}) \leq 2}$$

This is called the Clauser – Horne – Shimony – Holt (CHSH) inequality.

Quantum theory

We can compute each $\overbrace{E(\overline{A_i B_i})}^{\langle A_i B_i \rangle = \langle \psi | A_i B_i | \psi \rangle}$ quantum mechanically.

Let $|\psi\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$.

$$\begin{cases} A_1 = Z_1 = Z \otimes I \\ A_2 = X_1 = X \otimes I \end{cases}$$

$$\begin{cases} B_1 = \dfrac{-Z_2 - X_2}{\sqrt{2}} \\ B_2 = \dfrac{Z_2 - X_2}{\sqrt{2}} \end{cases}$$

$X_2 = I \otimes X$

$Z_2 = I \otimes Z$

$$E(\overline{A_1 B_1}) = \langle \psi | A_1 B_1 | \psi \rangle = \langle \psi | Z_1 \left(\frac{-Z_2 - X_2}{\sqrt{2}} \right) | \psi \rangle$$

$$= \underbrace{\left(\frac{\langle 01 | + \langle 10 |}{\sqrt{2}} \right)} \left(\frac{|01\rangle + |10\rangle}{2} \right) = \frac{1}{\sqrt{2}}$$

$$\langle A_1 B_2 \rangle = \langle A_2 B_1 \rangle = -\langle A_2 B_2 \rangle = 1/\sqrt{2}$$

Therefore

$$E(X) = \langle A_1 B_1 \rangle + \langle A_2 B_1 \rangle + \langle A_1 B_2 \rangle - \langle A_2 B_2 \rangle$$

$$= \frac{1}{\sqrt{2}} \left(1 + 1 + 1 - (-1) \right) = \underline{\underline{2\sqrt{2}}}$$

This violates the CHSH inequality

$$E(X) \le 2.$$

"Statistical predictions of QM cannot be reproduced by a local HVM."

<u>Picture</u>

$$\left(\{\pm 1\}^4, P_{HVM}\right) \leftarrow \text{HVM}$$

QM

$$\sim \left(\{\pm1\}^2_{A_1,B_1},\ P_e = P_{HVM}^{A_1B_1}\right) \qquad \left(\{\pm1\}^2_{A_1,B_2},\ P_e = P_{HVM}^{A_1B_2}\right) \sim$$

$$\sim \left(\{\pm1\}^2_{A_2,B_1},\ P_e = P_{HVM}^{A_2B_1}\right) \qquad \left(\{\pm1\}^2_{A_2,B_2},\ P_e = P_{HVM}^{A_2B_2}\right) \sim$$

Is there a joint prob. dist P_{HVM} such that it marginalizes to the q.m. prob dist on pairs? Ans: No.

- Gleason $\qquad \Pi \longmapsto Tr(\rho\Pi)$
- Kochen-Specker $\Leftarrow$ Gleason