Linear algebra in Dirac notation

1) Vector spaces: Let V be a (complex) vector space.

A non-zero vector $v \in V$ will be denoted by

$|v\rangle$: vector in ket notation

Ex $\mathbb{C}^n = \{ (z_1, \dots, z_n) \mid z_i \in \mathbb{C} \}$

vectors are denoted by

$$|v\rangle = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

Rem Zero vector is denoted by 0.

2) The span of $\{ |v_1\rangle, \dots, |v_n\rangle \}$

is the subspace of V given by

$$\text{span}\{ |v_1\rangle, \dots, |v_n\rangle \} = \{ |v\rangle \mid \sum_i a_i |v_i\rangle \quad a_i \in \mathbb{C} \}$$

If $\text{span}\{ |v_1\rangle, \dots, |v_n\rangle \} = V$ then

$\{ |v_1\rangle, \dots, |v_n\rangle \}$ is called a spanning set.

3) A set $\{|v_1\rangle, \dots, |v_n\rangle\}$ is linearly dependent if there exists $a_1, \dots, a_n \in \mathbb{C}$, not all 0, such that

$$\sum_i a_i |v_i\rangle = 0$$

4) A set of vectors is linearly independent if it is not linearly dependent.

Fact Any two linearly independent spanning sets have the same number of vectors.

5) Such a set is called a basis and this number is called the dimension of the vector space, denoted by $\dim(V)$.

We will only be concerned with finite-dimensional vector spaces.

Ex A basis for $\mathbb{C}^n$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

We will write

$$|e_i\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th}$$

For $\mathbb{C}^2$ we write $\{|0\rangle, |1\rangle\}$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Linear operators

1) A linear operator between two vector spaces V and W is a function

$$A : V \to W \qquad |v\rangle \mapsto A|v\rangle$$

such that

$$A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i \, A|v_i\rangle .$$

Ex Identity operator

$$I_V : V \to V \qquad |v\rangle \mapsto |v\rangle$$

Zero operator

$$O_V : V \to V \qquad |v\rangle \mapsto 0$$

We will write $L(V, W)$ for the set of linear operators $A : V \to W$.

Fact $L(V, W)$ is a complex vector space

$$(A + B)|v\rangle = A|v\rangle + B|v\rangle$$

$$(\alpha A)|v\rangle = \alpha A|v\rangle \qquad \alpha \in \mathbb{C}$$

2) Given U, V, W vector spaces we can define composition

$$L(U, V) \times L(W, U) \to L(W, V)$$

$$(A : U \to V,\ B : W \to U) \mapsto AB : W \to V$$
$$|w\rangle \mapsto A(B|w\rangle)$$

3) We say V and W are isomorphic if there exists $A: V \to W$ and $B: W \to V$ such that $AB = I_W$ and $BA = I_V$.

Notation: $V \cong W$

Ex Let V be vector space of dim n.

$\{|v_1\rangle, \dots, |v_n\rangle\}$ basis of V.

We can define $A \in L(V, \mathbb{C}^n)$

$$A|v\rangle = A\left(\sum_i a_i |v_i\rangle\right)$$

$$= \sum_i a_i \underbrace{A|v_i\rangle}_{\text{define to be } |e_i\rangle} \qquad |e_i\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th}$$

$$= \sum_i a_i |e_i\rangle$$

We can define $B \in L(\mathbb{C}^n, V)$

$$B|e_i\rangle = |v_i\rangle$$

Observe that $AB = I_{\mathbb{C}^n}$ and $BA = I_V$.

Therefore $V \cong \mathbb{C}^n$.

Matrix representation

Let $\{|v_1\rangle, \dots, |v_n\rangle\}$ basis for V

$\{|w_1\rangle, \dots, |w_m\rangle\}$ basis for W

1) A linear operator $A: V \to W$ can be represented as a $n \times m$ - matrix with entries A_{ij}:

$$A|v_j\rangle = \sum_i A_{ij} |w_i\rangle$$

Ex $\quad V = \mathbb{C}^2 \quad$ with $\{|0\rangle, |1\rangle\}$

- $X : \mathbb{C}^2 \to \mathbb{C}^2 \qquad |0\rangle \mapsto |1\rangle$, $\quad |1\rangle \mapsto |0\rangle$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- $Z : \mathbb{C}^2 \to \mathbb{C}^2 \qquad |0\rangle \mapsto |0\rangle$, $\quad |1\rangle \mapsto -|1\rangle$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $Y : \mathbb{C}^2 \to \mathbb{C}^2 \qquad |0\rangle \mapsto i|1\rangle$, $\quad |1\rangle \mapsto -i|0\rangle$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Note that $\quad Y = i X Z$

The matrices $\{I, X, Y, Z\}$ are called the Pauli matrices.

Alternate notation:

$$\sigma_0 = I \;,\quad \sigma_1 = \sigma_x = X \;,\quad \sigma_2 = \sigma_y = Y \;,\quad \sigma_3 = \sigma_z = Z$$

2) Composition of linear operators correspond to matrix multiplication

$B : V \to U$ and $A : U \to V$ we have

$$(AB)_{ij} = \sum_{k=1}^{\dim U} A_{ik} B_{kj}$$

Notation $\quad$ When $V = W$ we write $L(V) = L(V, V)$

Inner products

1) A function $(\cdot,\cdot) : V \times V \to \mathbb{C}$ is called an inner product if

a) $(\cdot,\cdot)$ is linear in the second argument

$$(|v\rangle, \sum \lambda_i |w_i\rangle) = \sum \lambda_i (|v\rangle, |w_i\rangle)$$

b) Conjugate symmetry

$$(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$$

Notation: For $z \in \mathbb{C}$ we write z^* for complex conjugate.

c) Positive definiteness

$(|v\rangle, |v\rangle) \geq 0$ with equality if and only if $|v\rangle = 0$.

Bra notion

For $v \in \mathbb{C}^n$ with $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ we write

$$v^* = (v_1^* \ \dots \ v_n^*)$$

We will denote v^* as follows:

$$\langle v| = (v_1^* \dots v_n^*)$$

Ex For $y, z \in \mathbb{C}^n$ define

$\{|e_i\rangle\}$

$$(y, z) = \sum_i y_i^* z_i \quad \in \mathbb{C}$$

This gives an inner product on $\mathbb{C}^n$.

In bra-ket notation (y, z) is denoted by

$$\langle y | z \rangle = (y_1^* \cdots y_n^*) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

2) A vector space V together with an inner product is called an inner product space.
Also called finite-dimensional Hilbert space.

For $v, w \in V$ we write

$$\langle v | w \rangle \quad \text{for} \quad (v, w)$$

3) Norm of $|v\rangle$ is defined by

$$\|v\| = \| \, |v\rangle \, \| = \sqrt{\langle v | v \rangle}$$

$|v\rangle$ is called normalized, or unit vector, if

$$\|v\| = 1$$

We can normalize a vector

$$|v\rangle \mapsto \frac{1}{\|v\|} |v\rangle \quad \text{(normalization)}$$

4) A set $\{|v_1\rangle, \ldots, |v_k\rangle\}$ of vectors is called

a) orthogonal if

$$\langle v_i | v_j \rangle = 0 \quad \forall \, i, j \quad i \neq j$$

b) orthonormal if

$$\langle v_i | v_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\forall \, i, j$$

Gram-Schmidt procedure

Let $\{|w_1\rangle, \dots, |w_n\rangle\}$ be a basis for an inner product space V.

The Gram-Schmidt process gives an orthonormal basis $\{|v_1\rangle, \dots, |v_n\rangle\}$ as follows:

a) $|v_1\rangle = \frac{1}{\||w_1\rangle\|} |w_1\rangle$

b) for $1 \le k \le n-1$

$$|v_{k+1}\rangle = \frac{|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle}{\left\| |w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i | w_{k+1}\rangle |v_i\rangle \right\|}$$

Convenient notation

1) Instead of writing $\{|v_1\rangle, \dots, |v_n\rangle\}$ for an orthonormal basis we will sometimes write $\{|1\rangle, \dots, |n\rangle\}$

a) $\{|v_i\rangle\}_{i=1}^{n}$ determines an isomorphism

$$V \cong \mathbb{C}^n \quad \text{and} \quad \{|i\rangle\}_{i=1}^{n} = \{|e_i\rangle\}_{i=1}^{n}$$

b) $\{|i\rangle\}_{i=1}^{n}$ the vectors $|i\rangle$ stand for vectors in V, such as $|v_i\rangle$.

" For $\mathbb{C}^2$ we write $\{|0\rangle, |1\rangle\}$ for $\{e_1, e_2\}$. "

With this notation

$$|v\rangle = \sum_{i=1}^{n} v_i \, |i\rangle$$

$$|w\rangle = \sum_{i=1}^{n} w_i \, |i\rangle$$

$$\langle w|v\rangle = \left(\sum w_j |j\rangle , \sum v_i |i\rangle \right)$$

$$= \sum_{i,j} w_j^* v_i \underbrace{\langle j|i\rangle}_{\delta_{ij}}$$

$$= \sum_i w_i^* v_i$$

$$= (w_1^* \dots w_n^*) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

2) We can think of $\langle v|$ as a linear operator in $L(V, \mathbb{C})$

$$\langle v| : V \longrightarrow \mathbb{C} \qquad |v'\rangle \mapsto \langle v|v'\rangle$$

More generally, for $v \in V$, $w \in W$

$|w\rangle\langle v|$ can be regarded as a linear operator

in $L(V, W)$:

$$|w\rangle\langle v| : V \longrightarrow W \qquad |v'\rangle \longmapsto \langle v|v'\rangle \; |w\rangle$$

This is called the outer product of $|v\rangle$ and $|w\rangle$.

We can take linear combinations of

$$\left\{ |w_j\rangle\langle v_i| \;\middle|\; 1 \le j \le m , 1 \le i \le n \right\}$$

to define $V \longrightarrow W$

$$\left(\sum_{i,j} a_{ij} \, |w_j\rangle\langle v_i| \right) |v'\rangle = \sum_{i,j} a_{ij} \, \langle v_i | v'\rangle \, |w_j\rangle$$

Ex 1) Consider $\mathbb{C}^n$ with $\{ |e_i\rangle \}$

$$\sum_i |e_i\rangle\langle e_i| : \mathbb{C}^n \to \mathbb{C}^n$$

$$\left(\sum_i |e_i\rangle\langle e_i| \right) |v\rangle = \sum_i \underbrace{\langle e_i | v\rangle}_{v_i} |e_i\rangle = |v\rangle$$

Therefore $\sum_i |e_i\rangle\langle e_i| = I_{\mathbb{C}^n}$

This is called the completeness relation.

2) Let $\{ |v_i\rangle \}$ and $\{ |w_j\rangle \}$ be orthonormal bases for V and W, and $A : V \to W$.

$$A = I_W \, A \, I_V$$

$$= \sum_j |w_j\rangle\langle w_j| \; A \; \sum_i |v_i\rangle\langle v_i|$$

$$(\cdots w_j \cdots) \begin{pmatrix} & \vdots & \\ \cdots & A_{ij} & \cdots \\ & \vdots & \end{pmatrix} \begin{pmatrix} \vdots \\ v_i \\ \vdots \end{pmatrix}$$

$$= \sum_{i,j} \underbrace{\langle w_j | A | v_i \rangle}_{A_{ij}} |w_j\rangle\langle v_i|$$

$$\langle w_j | A | v_i \rangle = \left(|w_j\rangle, A|v_i\rangle \right)$$

This is a typical application of completeness relation.

Cauchy - Schwarz inequality

Let $v, w \in V$. Then

$$\|v\| = \sqrt{\langle v | v \rangle}$$
$$\|w\| = \sqrt{\langle w | w \rangle}$$

$$|\langle v | w \rangle| \leq \|v\| \cdot \|w\|$$

with equality if and only if one of v, w is a scalar multiple of the other.

Proof Using Gram - Schmidt procedure we can construct an orthonormal basis

$$\{|v_i\rangle\}_{i=1}^{n} \quad \text{such that}$$

$$|v_1\rangle = \frac{1}{\|w\|} |w\rangle$$

Then

$$\langle v | v \rangle \langle w | w \rangle = \sum_i \langle v | v_i \rangle \langle v_i | v \rangle \langle w | w \rangle$$

$$\geq \langle v | v_1 \rangle \langle v_1 | v \rangle \langle w | w \rangle$$

$$= \langle v | w \rangle \langle w | v \rangle \frac{\langle w | w \rangle}{\|w\|^2}$$

$$= |\langle v | w \rangle|^2 \qquad \blacksquare$$

Eigenvectors and eigenvalues

1) An eigenvector of a linear operator $A \in L(V)$ is a non-zero vector $|v\rangle$ such that

$$A|v\rangle = \lambda |v\rangle \quad \text{for some } \lambda \in \mathbb{C}.$$

λ is called the eigenvalue corresponding to $|v\rangle$

We will write V_λ for the eigenspace corresponding to λ:

$$V_\lambda = \{ v \in V \mid A|v\rangle = \lambda |v\rangle \}$$

λ is said to be degenerate if $\dim V_\lambda > 1$.

Fact: Every linear operator has at least one eigenvalue and a corresponding eigenvector since the characteristic polynomial

$$c(\lambda) = \det(A - \lambda I_V)$$

has at least one root.

2) A is diagonalizable if

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

where $|i\rangle$ form an orthonormal set of eigenvectors with corresponding eigenvalues λ_i.

Ex

1) $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

2) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad X: \begin{array}{l} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle \end{array}$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$X|+\rangle = |+\rangle, \quad X|-\rangle = -|-\rangle$$

$$X = |+\rangle\langle +| - |-\rangle\langle -|$$

Adjoints

1) The adjoint (Hermitian conjugate) of $A \in L(V)$ is the linear operator $A^\dagger \in L(V)$ defined by the equation

$$(|v\rangle, A|w\rangle) = (A^\dagger|v\rangle, |w\rangle)$$

for all $v, w \in V$.

$$\langle v|A|w\rangle = \langle v|(A^\dagger)^\dagger|w\rangle$$

In matrix representation

$$A^\dagger = (A^*)^T \quad \text{i.e. transpose of the conjugate.}$$

By convention $|v\rangle^\dagger = \langle v|$

Important classes of operators

1) Normal operators : $A \in L(V)$ is normal if

$$A A^{\dagger} = A^{\dagger} A$$

We will see that normal operators are diagonalizable

2) Hermitian operators : $A \in L(V)$ is Hermitian if $A = A^{\dagger}$. Note that Hermitian operators are normal.

Notation : $\text{Herm}(V)$

3) Positive operators (positive semidefinite) :

$A \in L(V)$ is positive if

$$\langle v | A | v \rangle \in \mathbb{R}_{\geq 0} \quad \text{(real and non-negative)}$$

for all $v \in V$

Notation $\text{Pos}(V)$

4) Positive definite operators

$A \in L(V)$ is positive definite if

$$\langle v | A | v \rangle \in \mathbb{R}_{> 0} \quad \text{(real and positive)}$$

for all non-zero $v \in V$.

Notation $\text{Pd}(V)$

Rem (3) & (4) are Hermitian.

$$\mathrm{Pos}(V) = \{ B^{+}B \mid B \in L(V) \}$$

$$\mathrm{Pd}(V) = \{ A \in \mathrm{Pos}(V) \mid \det A \neq 0 \}$$

5) Projection operators: $P \in \mathrm{Pos}(V)$ is a projection operator if $P^2 = P$.

Notation $\mathrm{Proj}(V)$

The image of P will be denoted by V_P.

More: a) Given $W \subset V$ subspace we can construct a projector P with $\mathrm{im}(P) = W$,

Let $\{ |i\rangle \}_{i=1}^{\dim(W)}$ orthonormal basis for W.

Extend this to orthonormal basis for V

$$\{ |i\rangle \}_{i=1}^{\dim(V)}$$

Define

$$P = \sum_{i=1}^{\dim W} |i\rangle\langle i|$$

$$P|v\rangle = \left(\sum_{i=1}^{\dim W} |i\rangle\langle i| \right) \left(\sum_{j=1}^{\dim V} v_j |j\rangle \right)$$

$$= \sum_{i=1}^{\dim W} v_i |i\rangle \in W$$

Therefore $\mathrm{im}(P) = W$. $(W = V_P)$

b) Conversely given P we can diagonalize

$$P = \sum_{i=1}^{\dim V} \lambda_i |i\rangle\langle i|$$

Since $P^2 = P$ and $\{|i\rangle\}$ orthonormal we have $\lambda_i = 0$ or 1.

$$\left(P^2 = \sum_i \lambda_i^2 |i\rangle\langle i| \right)$$

$$\lambda_i \geq 0$$

Therefore $P = \sum_i |i\rangle\langle i|$

This gives a bijection

$$\mathrm{Proj}(V) \longrightarrow \{ W \subset V \mid W \text{ subspace} \}$$

$$P \longmapsto V_P$$

The orthogonal complement of P is the projector

$$Q = I_V - P = \sum_{i=\dim(V_P)+1}^{\dim V} |i\rangle\langle i|$$

V_Q is called the orthogonal complement of V_P.

6) Unitary operators: $U \in L(V)$ is unitary if $U^\dagger U = I_V$.

Notation $U(V)$

Rem Note that

$U^\dagger U = I_V$ implies that $\det U \neq 0$ and therefore $U^{-1} = U^\dagger$. This implies that $U U^\dagger = I_V$ as well.

We have

$$(U|v\rangle, U|w\rangle) = \langle v| U^\dagger U |w\rangle$$
$$= \langle v|w\rangle \qquad \forall\, v,w \in V.$$

Every $U \in U(V)$ is diagonalizable and its eigenvalues λ satisfy $|\lambda| = 1$.

Rem: The adjoint can also be defined for $A \in L(V,W)$ similarly by the equation

$$(w, Av) = (A^\dagger w, v)$$
$$\forall\, v \in V,\ w \in W.$$

An operator $A \in L(V,W)$ is an isometry if $A^\dagger A = I_V$.

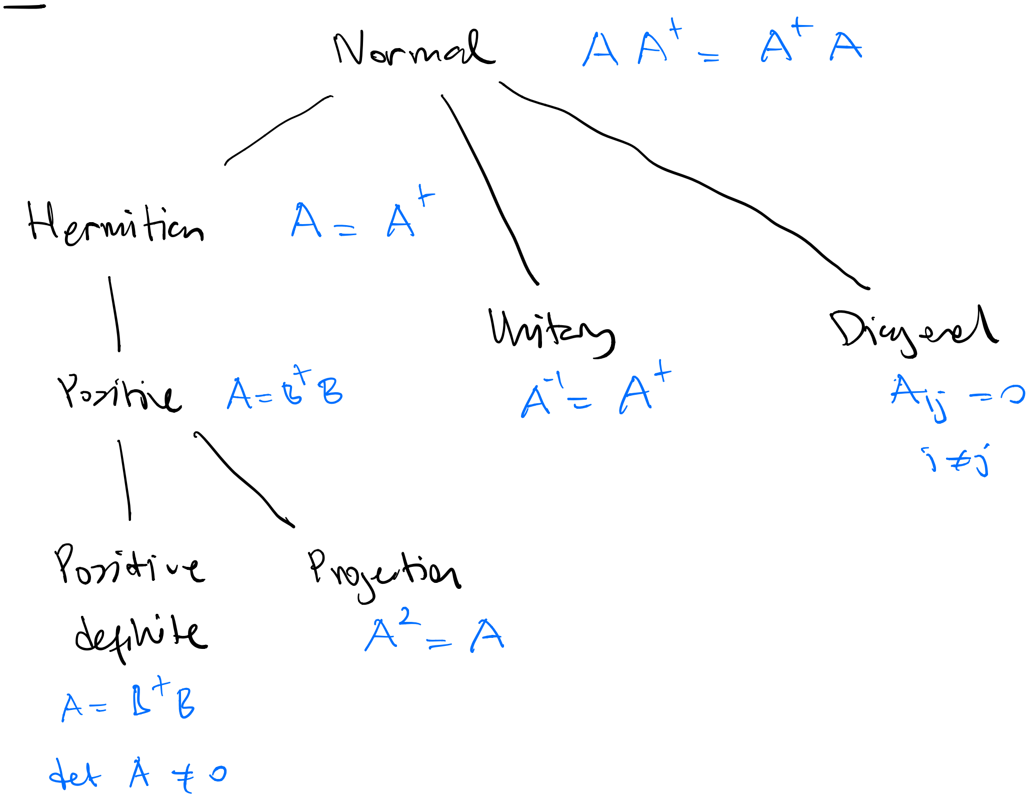
Notation $U(V,W)$. Note that $U(V,V) = U(V)$.

7) Diagonal operators: $A \in L(\mathbb{C}^n)$ is diagonal if $A_{ij} = 0$ for all $1 \leq i, j \leq n$ and $i \neq j$.

Rem $A \in L(\mathbb{C}^n)$ is diagonalizable if and only if there exists $U \in U(\mathbb{C}^n)$ such that

$$U A U^{+} \text{ is diagonal.}$$

Picture



Theorem (Spectral decomposition)

$A \in L(V)$ is diagonalizable if and only if A is normal.

Proof (Easy direction $\Longrightarrow$)

If A is diagonalizable then

$$A = \sum_i \lambda_i |v_i\rangle\langle v_i|$$

where $\{|v_i\rangle\}$ is an orthonormal basis consisting of eigenvectors.

Then

$$A^\dagger A = \left(\sum_j \lambda_j^* |v_j\rangle\langle v_j|\right)\left(\sum_i \lambda_i |v_i\rangle\langle v_i|\right)$$

$$= \sum_{i,j} \lambda_j^* \lambda_i \underbrace{\langle v_j | v_i\rangle}_{\delta_{ji}} |v_j\rangle\langle v_i|$$

$$= \sum_i \lambda_i^* \lambda_i |v_i\rangle\langle v_i|$$

$$= \sum_i \lambda_i \lambda_i^* |v_i\rangle\langle v_i|$$

$$= \sum_{i,j} \lambda_i \lambda_j^* \underbrace{\langle v_i | v_j\rangle}_{\delta_{ij}} |v_i\rangle\langle v_j|$$

$$= \left(\sum_i \lambda_i |v_i\rangle\langle v_i| \right) \left(\sum_j \lambda_j^* |v_j\rangle\langle v_j| \right)$$

$$= A A^\dagger$$

($\Leftarrow$)

We will do induction on $d = \dim V$.
For $d = 1$ result is clear: Every $A \in L(V)$ is diagonalizable and normal.

Let λ be an eigenvalue of A.
Let P be the projector onto the eigenspace V_λ ($= V_P$).
Let $Q = I_V - P$. $\rightarrow$ $I_V = P + Q$

Then

$$A = I_V A I_V$$

$$= PAP + \underbrace{QAP + PAQ}_{\text{Goal: } = 0} + QAQ$$

1) $QAP = 0$

$$AP|v\rangle = \lambda P|v\rangle \in V_P = V_\lambda$$

$$QAP|v\rangle = \lambda QP|v\rangle = 0$$

2) $PAQ = 0$

Let $w \in V_P = V_\lambda$

$$AA^\dagger |w\rangle \underset{\text{(normality)}}{=} A^\dagger A\, |w\rangle = \lambda A^\dagger |w\rangle$$

Therefore $A^\dagger |w\rangle \in V_\lambda = V_P$

Similar to (1) we have

$$QA^\dagger P = 0 \qquad (2*)$$

$$\underbrace{Q A^\dagger \underbrace{P\, |w\rangle}_{V_P = V_\lambda}}_{V_P} = 0$$

Take the adjoint of $(2*)$

$$(QA^\dagger P)^\dagger = 0 \quad \sim \quad P^\dagger (A^\dagger)^\dagger Q^\dagger = 0$$

$$PAQ = 0$$

$$PAQ = 0$$

Combining (1) & (2)

$$A = PAP + QAQ$$

3) QAQ is normal.

By (1) & (2*)

$$QA = QA(P+Q) \overset{(1)}{=} QAQ \quad (3.1)$$

$$QA^{+} = QA^{+}(P+Q) = QA^{+}Q \quad (3.2)$$

$$\begin{aligned}(QAQ)(QA^{+}Q) &\overset{3.1}{=} QA\,QA^{+}Q \\ &\overset{3.2}{=} QA\,A^{+}Q \\ &= QA^{+}AQ \\ &\overset{3.2}{=} QA^{+}Q\,AQ \\ &\underset{(Q=Q^2)}{=} (QA^{+}Q)(QAQ)\end{aligned}$$

Here $QAQ \in L(V_Q)$ where $\dim V_Q < d$

By induction QAQ is diagonalizable, say with respect to $\{|w_i\rangle\}_{i=1}^{\dim V_Q}$

Let $\{|v_i\rangle\}_{i=1}^{\dim V_P}$ an orthonormal basis for V_P.

Then A is diagonal in

$$\{|v_i\rangle, |w_j\rangle\}_{i,j} \quad \blacksquare$$

Tensor product

1) Given vector spaces V and W the tensor product, denoted by $V \otimes W$, is the vector space with basis given by the symbols

$$|v\rangle \otimes |w\rangle \qquad v \in V \quad \text{and} \quad w \in W$$

modulo the relations

a) $$z\left(|v\rangle \otimes |w\rangle\right) = \left(z|v\rangle\right) \otimes |w\rangle$$
$$= |v\rangle \otimes \left(z|w\rangle\right)$$

b) $$\left(|v_1\rangle + |v_2\rangle\right) \otimes |w\rangle$$
$$= |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$$

c) $$|v\rangle \otimes \left(|w_1\rangle + |w_2\rangle\right)$$
$$= |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$$

i.e. $V \otimes W$ is the quotient vector space obtained this way.

2) Let $\{|v_i\rangle\}_i$ and $\{|w_j\rangle\}_j$ bases for V and W.

Then $V \otimes W$ has basis

$$\{|v_i\rangle \otimes |w_j\rangle\}_{i,j}$$

since

$$|v\rangle \otimes |w\rangle = \left(\sum_i a_i |v_i\rangle\right) \otimes \left(\sum_j b_j |w_j\rangle\right)$$

$$\overset{(a,b,c)}{=} \sum_{i,j} a_i b_j \; |v_i\rangle \otimes |w_j\rangle .$$

Notation: We will also write $|v\rangle|w\rangle$, $|vw\rangle$, $|v,w\rangle$ for $|v\rangle \otimes |w\rangle$.

Ex: a) $V = W = \mathbb{C}^2$ with $\{|0\rangle, |1\rangle\}$

$V \otimes W$ has basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

Rem: $\dim V = n$ and $\dim W = m$

Then $\dim(V \otimes W) = n \cdot m$

b) $V \otimes V \otimes V$ with $V = \mathbb{C}^2$

has basis $\{ |000\rangle, |001\rangle, |011\rangle, \dots \}$

3) Given $A \in L(V, W)$ and $B \in L(V', W')$ we can define

$$(A \otimes B)(\underbrace{|v\rangle}_{\in V} \otimes \underbrace{|v'\rangle}_{\in V'}) = \underbrace{A|v\rangle}_{\in W} \otimes \underbrace{B|v'\rangle}_{\in W'}$$

and obtain $A \otimes B \in L(V \otimes V', W \otimes W')$

by extending linearly.

This defines a linear operator

$$L(V, W) \otimes L(V', W') \longrightarrow L(V \otimes V', W \otimes W')$$

which turns out to be an isomorphism.

That is any linear operator

$C \in L(V \otimes V', W \otimes W')$ can be expressed as

$$C = \sum_i c_i \, A_i \otimes B_i$$

for some $A_i \in L(V, W)$, $B_i \in L(V', W')$

Ex a) $X : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ $\qquad$ $|0\rangle \longmapsto |1\rangle$

$|1\rangle \longmapsto |0\rangle$

$X \otimes X : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$

$$X \otimes X \left(|0\rangle \otimes |0\rangle\right) = X|0\rangle \otimes X|0\rangle$$

$$= |1\rangle \otimes |1\rangle$$

$$\begin{array}{ccc} |00\rangle & & |11\rangle \\ |01\rangle & \xmapsto{X \otimes X} & |10\rangle \\ |10\rangle & & |01\rangle \\ |11\rangle & & |00\rangle \end{array}$$

b) $Z : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ $\qquad$ $|0\rangle \longmapsto |0\rangle$

$|1\rangle \longmapsto -|1\rangle$

$Z \otimes X : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$

$$\begin{array}{ccc} |00\rangle & & |01\rangle \\ |01\rangle & \longmapsto & |00\rangle \\ |10\rangle & & -|11\rangle \\ |11\rangle & & -|10\rangle \end{array}$$

"unit complex numbers are also called phase"

Remark i) $Z \otimes X$ is not the same as ZX.

$Z \otimes X : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$

$ZX : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

ii) $Z \otimes X \neq X \otimes Z$.

4) Let V and W be inner product spaces.

We can define an inner product on $V \otimes W$:

$$\left(\sum_i a_i |v_i w_i\rangle, \sum_j b_j |v_j' w_j'\rangle \right) =$$

$$\sum_{i,j} a_i^* b_j \langle v_i | v_j' \rangle \langle w_i | w_j' \rangle$$

$v_i, v_j' \in V$ and $w_i, w_j' \in W$

Ex $V = W = \mathbb{C}^2$ with $\{|0\rangle, |1\rangle\}$

$x, y \in V \otimes W$

$$\langle x | y \rangle = \left(x_{00}^* \langle 00| + x_{01}^* \langle 01| + x_{10}^* \langle 10| + x_{11}^* \langle 11| \right)$$
$$\left(y_{00} |00\rangle + y_{01} |01\rangle + y_{10} |10\rangle + y_{11} |11\rangle \right)$$

$$= x_{00}^* y_{00} + x_{01}^* y_{01} + x_{10}^* y_{10} + x_{11}^* y_{11}.$$

5) Matrix representation of $A \otimes B$:

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & & \vdots \\ B_{p1} & \cdots & B_{pq} \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{A_{11} B} & \cdots & \boxed{A_{1n} B} \\ \vdots & & \vdots \\ \boxed{A_{m1} B} & \cdots & \boxed{A_{mn} B} \end{pmatrix}$$

$$\begin{pmatrix} A_{11} B_{11} & \cdots & A_{11} B_{1q} \\ \vdots & & \vdots \\ A_{11} B_{p1} & \cdots & A_{11} B_{pq} \end{pmatrix}$$

This is called the Kronecker product.

Ex

$$X \otimes Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

similarly for the other basis vectors.

Operator functions

1) Let A be a normal operator and $f: \mathbb{C} \to \mathbb{C}$ be a function.

By $f(A)$ we mean

$$f(A) = \sum_i f(\lambda_i) |v_i\rangle\langle v_i|$$

where $\{|v_i\rangle\}$ is an orthonormal basis of eigenvectors.

Ex a) $\exp(\alpha X) = \exp\left(\alpha\left(|+\rangle\langle+| - |-\rangle\langle-|\right)\right)$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $= e^{\alpha} |+\rangle\langle+| + e^{-\alpha} |-\rangle\langle-|$

$|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

$|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$

$\alpha \in \mathbb{C}$.

b) Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ unit vector.

Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$.

$\sigma_1 = X$

$\sigma_2 = Y$

$\sigma_3 = Z$

We write

$$v \cdot \sigma = \sum_{i=1}^{3} v_i \sigma_i$$

$$\exp(i\theta\, v \cdot \sigma) = \cos\theta\, I + i \sin\theta\, v \cdot \sigma$$

$\theta \in \mathbb{R}$

Idea: 1) Since v unit vector $v \cdot \sigma$ has eigenvalues ± 1.

2) Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

No need to find eigenvectors.

2) Trace of a matrix A is defined by

$$\operatorname{Tr} A = \sum_i A_{ii}$$

2.1) It satisfies the cyclicity property

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$$

Using this we can define the trace of a linear operator $A \in L(V)$ by

$$\operatorname{Tr}(A) = \underbrace{A_{ii}}$$

(A_{ij}) matrix represent'n for A.

Note that any two matrix representation A and A' are related by $A' = BAB^{-1}$ so that

$$\operatorname{Tr}(A') = \operatorname{Tr}((BA)(B^{-1})) = \operatorname{Tr}(B^{-1}BA) = \operatorname{Tr}(A).$$

2.2) Trace is the unique linear operator

$$\mathrm{Tr}: L(V) \longrightarrow \mathbb{C} \quad \text{such that}$$

$$\mathrm{Tr}(|v\rangle\langle u|) = \langle u|v\rangle .$$

$$|v\rangle\langle u| : V \longrightarrow V$$
$$|v'\rangle \longmapsto \langle u|v'\rangle |v\rangle$$

2.3) We can define an inner product on $L(V)$

$$(A, B) = \mathrm{Tr}(A^{\dagger} B)$$

called the trace inner product.

Commutator and anti-commutator

The commutator of A and B is defined by

$$[A, B] = AB - BA$$

i.e. $[A, B] = 0$ if and only if $AB = BA$.
In this case we say A commutes with B.

The anti-commutator of A and B:

$$\{A, B\} = AB + BA$$

Ex: $[\sigma_i, \sigma_i] = 0$
$\{\sigma_i, \sigma_j\} = 0 \quad i \neq j$

Theorem (Simultaneous diagonalization) $[A,B] = AB - BA$

Let A and B be Hermitian operators.
Then $[A,B]=0$ if and only if A and B are simultaneously diagonalizable i.e. there exists an orthonormal basis in which A and B are both diagonal.

Proof ($\Leftarrow$)

$$[A,B] = \Big[\sum_i \lambda_i^A |v_i\rangle\langle v_i| \,,\, \sum_j \lambda_j^B |v_j\rangle\langle v_j|\Big]$$

$$\begin{pmatrix} \lambda_1^A & & 0 \\ & \ddots & \\ 0 & & \lambda_n^A \end{pmatrix} \qquad \begin{pmatrix} \lambda_1^B & & 0 \\ & \ddots & \\ 0 & & \lambda_n^B \end{pmatrix}$$

$$= \sum_i \lambda_i^A \lambda_i^B |v_i\rangle\langle v_i| - \sum_i \lambda_i^B \lambda_i^A |v_i\rangle\langle v_i|$$

$$= 0.$$

($\Rightarrow$) a) Key idea: if λ is an eigenvalue of A and V_λ eigenspace then

$$B|v\rangle \in V_\lambda \qquad \forall\, |v\rangle \in V_\lambda$$

More explicitly, let $\{|\lambda, j\rangle\}_{j=1}^{\dim V_\lambda}$ be an orthonormal basis for V_λ. Then

$$A(B|\lambda,j\rangle) = \underbrace{BA|\lambda,j\rangle}_{\lambda|\lambda,j\rangle} = \lambda \underbrace{(B|\lambda,j\rangle)}_{\in V_\lambda}$$

b) Let P_λ be the projector onto V_λ.
Let $B_\lambda = P_\lambda B P_\lambda$.
We have

$$(B_\lambda)^+ = P_\lambda B^+ P_\lambda = P_\lambda B P_\lambda = B_\lambda$$

Therefore B_λ is diagonalizable. For an eigenvalue p of B_λ let $V_{\lambda,p}$ for the corresponding eigenspace in V_λ.
Let $\{|\lambda,p,k\rangle\}_k$ be an orthonormal basis of $V_{\lambda,p}$.

c) Note that

$$\underbrace{B_\lambda |\lambda,p,k\rangle}_{p|\lambda,p,k\rangle} = P_\lambda B P_\lambda |\lambda,p,k\rangle$$

$$= P_\lambda B |\lambda,p,k\rangle$$

$$= B |\lambda,p,k\rangle$$

then

$$B|\lambda,p,k\rangle = p|\lambda,p,k\rangle.$$

Therefore $\{|\lambda, p_{(\lambda)}, k_{(\lambda,p)}\rangle\}_{\lambda,p,k}$ is an orthonormal set of eigenvectors of both A and B. ☒

$V \quad V_\lambda \quad V_{\lambda,p} \quad \{|\lambda,p,k\rangle\}$

$A \quad B$

Ex $[X\otimes X, Z\otimes Z] = 0$

$Z\otimes Z$ is diagonal in

	λ
$\vert 00\rangle$	1
$\vert 01\rangle$	-1
$\vert 10\rangle$	-1
$\vert 11\rangle$	1

$V_1 \quad \{\,|00\rangle, |11\rangle\,\} \qquad \left\{ \frac{|00\rangle+|11\rangle}{\sqrt{2}}, \frac{|00\rangle-|11\rangle}{\sqrt{2}} \right\}$

$V_{-1} \quad \{\,|01\rangle, |10\rangle\,\} \qquad \left\{ \frac{|01\rangle+|10\rangle}{\sqrt{2}}, \frac{|01\rangle-|10\rangle}{\sqrt{2}} \right\}$

$$\left\{ \frac{|00\rangle+|11\rangle}{\sqrt{2}}, \frac{|00\rangle-|11\rangle}{\sqrt{2}}, \frac{|01\rangle+|10\rangle}{\sqrt{2}}, \frac{|01\rangle-|10\rangle}{\sqrt{2}} \right\}$$

diagonalize both $Z\otimes Z$ and $X\otimes X$.

$V_\lambda \qquad V_{\lambda,\mu} \qquad |\lambda,\mu\rangle$

$V_1 \qquad V_{1,1} \leftarrow$ 1-dim

$|1,1\rangle$

(Recall $X : \mathbb{C}^2 \to \mathbb{C}^2$ $\quad |0\rangle \mapsto |1\rangle$, $\quad |1\rangle \mapsto |0\rangle$

$|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \qquad |-\rangle = \frac{|0\rangle-|1\rangle}{2}$)

$$X\otimes X\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right) = X\otimes X\left(\frac{|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle}{\sqrt{2}}\right)$$

$$= \frac{(X|0\rangle)\otimes(X|0\rangle) + (X|1\rangle)\otimes(X|1\rangle)}{\sqrt{2}}$$

$$= \frac{|1\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle}{\sqrt{2}}$$

$$= \frac{|11\rangle + |00\rangle}{\sqrt{2}}$$

Do this with matrices

$$V_1 = \operatorname{span}\{ |00\rangle, |11\rangle \}$$

matrix of $X \otimes X$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$(X \otimes X : V_1 \to V_1)$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |\tilde{+}\rangle\langle\tilde{+}| - |\tilde{-}\rangle\langle\tilde{-}|$$

$$|\tilde{+}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\tilde{-}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$V_1 \xrightarrow{\varphi} \mathbb{C}^2$$

$$|00\rangle \longmapsto |0\rangle$$

$$|11\rangle \qquad |1\rangle$$

$$X \otimes X \longmapsto X$$

$$\begin{tikzcd} V_1 \arrow[r, "\varphi"] \arrow[d, "X\otimes X"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{C}^2 \arrow[d, "X"] \\ V_1 \arrow[r, "\varphi"] & \mathbb{C}^2 \end{tikzcd}$$

Theorem (Polar decomposition)
Let $A \in L(V)$.

Then there exists $U \in U(V)$ and $J, K \in Pos(V)$ such that

$$A = \underbrace{UJ}_{\text{left polar decomposition}} = \underbrace{KU}_{\text{right polar decomposition}}$$

"Consider $V = \mathbb{C}$
$L(V) = \mathbb{C}$ "

where $J = \sqrt{A^+A}$ and $K = \sqrt{AA^+}$.
If A is invertible then U is unique.

Rem Here $A^+A \in Pos(V)$. $J = \sqrt{A^+A}$ is the unique positive operator such that

$$J^2 = A^+A. \quad (\text{similarly for } K)$$

Proof a) $J = \sqrt{A^+A}$ is positive so we can write

$$\boxed{J = \sum_i \lambda_i |i\rangle\langle i|} \qquad \lambda_i \in \mathbb{R}_{\geq 0}$$

For $\lambda_i \neq 0$ define

$|i\rangle$: eigenvector of J corresponding to λ_i.

$$|v_i\rangle = \frac{1}{\lambda_i} A |i\rangle$$

$\{|v_i\rangle\}$ is an orthonormal set:

$$\langle v_j|v_i\rangle = \langle j|\, A^\dagger \frac{1}{\lambda_j}\,\frac{1}{\lambda_i}\, A\,|i\rangle$$

$$= \frac{1}{\lambda_j\lambda_i}\,\underbrace{\langle j|\,\underbrace{A^\dagger A}_{J^2}\,|i\rangle}_{\underbrace{\underbrace{\langle j|J}_{\lambda_j\langle j|}\ \underbrace{J|i\rangle}_{\lambda_i|i\rangle}}_{\lambda_j\lambda_i\langle j|i\rangle}} = \delta_{ji}$$

We can extend this set to an orthonormal basis $\{|v_i\rangle\}$ of V.

Let $\quad U := \sum_i |v_i\rangle\langle i|$.

b) b.1) Note that for $\lambda_i \neq 0$

$$UJ\,|i\rangle = \Big(\sum_k |v_k\rangle \overbrace{\langle k|\Big)\Big(\sum_j \lambda_j\, |j\rangle}^{\delta_{kj}} \overbrace{\langle j|\Big)\,|i\rangle}^{\delta_{ij}}$$

$$= \lambda_i \underbrace{|v_i\rangle}_{\frac{1}{\lambda_i}A|i\rangle} = A\,|i\rangle$$

b.2) For $\lambda_i = 0$

$$UJ\,|i\rangle = U\,\overbrace{\Big(\sum_j \lambda_j\, |j\rangle\langle j|\Big)\,|i\rangle}^{\lambda_i|i\rangle} = 0$$

Note $A|i\rangle = 0 \leftarrow$ since

$$\|A|i\rangle\| = \langle i|\underbrace{A^\dagger A}_{J^2}|i\rangle = \lambda_i^2 = 0.$$

Therefore $A = UJ$.

c) If A is invertible then

$$0 \neq \det A = \det(UJ) = \det U \cdot \det J \neq 0$$

J is invertible:

$$A = UJ \Rightarrow U = AJ^{-1}$$ ☒

$\boxed{\text{Normal}}$ A is diag $\iff$ A is normal.

"$V = \mathbb{C}$"

Normal

Herm

Pos

Pd

$\boxed{\text{Unitary}}$ is a group: if A and $B \in U(V)$ then $AB \in U(V)$.

$A, B \in \text{Herm}(V)$

$$(AB)^\dagger = B^\dagger A^\dagger = BA$$

Corollary (Singular value decomposition)

Let A square matrix.
Then there exists unitary matrices U and V, a diagonal D with non-negative entries such that

$$A = UDV.$$

Proof By the polar decomposition

$$A = WJ$$

where W: unitary, $J = \sqrt{A^+A}$.

Spectral decomposition:

$$J = TDT^+$$

where T: unitary.

Then we have

$$A = W(TDT^+)$$

and we take

$$U = WT \quad \text{and} \quad V = T^+$$ $\boxtimes$

Jargon: The diagonal entries in D are called the singular values of A.