MATH 9144A - ASSIGNMENT 3

(1) (a) Let k be a commutative ring, and denote $\otimes_k$ by $\otimes$. Let R and S be k-algebras. Show that there is a natural map

$$\mathrm{Tor}_i^R(A,B)\otimes \mathrm{Tor}_j^S(A',B')\to \mathrm{Tor}_{i+j}^{R\otimes S}(A\otimes A', B\otimes B')$$

for R-modules A, B and S-modules A', B'. When R is commutative and augmented $\epsilon : R\to k$ show that $\mathrm{Tor}_*^R(k,k)$ is graded commutative $ab=(-1)^{|a||b|}ba$. (In fact it is a graded commutative k-algebra.)

(b) For $R=k[x]$ show that $\mathrm{Tor}_*^R(k,k)$ is the exterior algebra $\Lambda(y)$ where $|y|=1$.

(2) (a) Show that there is a natural map

$$\mathrm{Ext}_R^i(A,B)\otimes \mathrm{Ext}_S^j(A',B')\to \mathrm{Ext}_{R\otimes S}^{i+j}(A\otimes A', B\otimes B').$$

Show that if R is augmented then $\mathrm{Ext}_R^*(k,k)$ is graded commutative (k-algebra).

(b) Let p be a prime. For $k=\mathbb{F}_p$, and $R=k[\mathbb{Z}/p]$ show that

$$\mathrm{Ext}_R^*(k,k)=\begin{cases} k[x] & p=2\\ k[y]\otimes\Lambda(x) & p>2\end{cases}$$

as graded rings where $|x|=1$ and $|y|=2$.

(3) Consider the category $\mathbf{C}: 0\to 01\leftarrow 1$. Let $\underline{\mathbb{Z}}:\mathbf{C}\to\mathbf{Ab}$ denote the constant functor at $\mathbb{Z}$. Compute $(R^i\lim)(\underline{\mathbb{Z}})$. *Hint:* Regard $R^i\lim$ as an Ext functor in the functor category $\mathbf{Ab}^{\mathbf{C}}$.

(4) Let $H\subset G$ be a subgroup of finite index. Consider $\mathbb{Z}$ as a $\mathbb{Z}G$-module with trivial G-action. Prove that if $\mathrm{pd}_{\mathbb{Z}G}\mathbb{Z}<\infty$ then $\mathrm{pd}_{\mathbb{Z}G}\mathbb{Z}=\mathrm{pd}_{\mathbb{Z}H}\mathbb{Z}$. *Hint:* Use Shapiro's lemma.