# MATH 9144A - ASSIGNMENT 2

(1) A left $\mathbb{Z}G$-module $M$ can be regarded as a right module by $mg = g^{-1}m$. Therefore we can define the tensor product $M \otimes_{\mathbb{Z}G} N$ of two left $\mathbb{Z}G$-modules $M, N$.

   (a) Prove that $M \otimes_{\mathbb{Z}} N$ has the structure of a $\mathbb{Z}G$-module defined by the diagonal action $g(m \otimes n) = gm \otimes gn$. Deduce that $M \otimes_{\mathbb{Z}G} N = (M \otimes_{\mathbb{Z}} N)_G$, where $(-)_G$ denotes the coinvariant submodule.

   (b) Prove that $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ has the structure of a $\mathbb{Z}G$-module defined by the conjugation action $(gf)(m) = gf(g^{-1}m)$. Deduce that $\mathrm{Hom}_{\mathbb{Z}G}(M, N) = \mathrm{Hom}_{\mathbb{Z}}(M, N)^G$, where $(-)^G$ denotes the invariant submodule.

   (c) Let $P$ be a projective $\mathbb{Z}G$-module and $M$ be a $\mathbb{Z}$-free $\mathbb{Z}G$-module. Prove that $P \otimes_{\mathbb{Z}} M$ is $\mathbb{Z}G$-projective.

(2) Let $G$ be a finite group and $N : M \to M$ denote the norm map defined by multiplication by $\sum_{g \in G} g$. Prove that the induced map $\bar{N} : M_G \to M^G$ is an isomorphism if $M$ is a projective $\mathbb{Z}G$-module.

(3) Let $F$ be a right exact functor. Assume $\cdots \to C_1 \to C_0 \to M \to 0$ is exact where each $C_i$ satisfies $L_nF(C_i) = 0$ for $n > 0$. Prove that $L_nF(M) \cong H_n(F(C))$ for all $n$.

(4) Let $R = \mathbb{Z}/m$ and $M = \mathbb{Z}/d$ where $d > 1$ divides $m$. Prove that $M$ is <u>not</u> injective as an $R$-module if there exists a prime dividing both $d$ and $m/d$.

(5) Prove that the following are equivalent.

   (a) $N$ is an injective $R$-module.

   (b) $\mathrm{Ext}^i(M, N)$ vanishes for all $i > 0$ and all $M$.

   (c) $\mathrm{Ext}^1(M, N)$ vanishes for all $M$.

   State and prove the dual statement for the projective case.

(6) Prove that

   (a) $\mathrm{Tor}_i^R(A, \oplus_k B_k) \cong \oplus_k \mathrm{Tor}_i^R(A, B_k)$.

   (b) $\mathrm{Ext}_R^i(\oplus_k A_k, B) \cong \prod_k \mathrm{Ext}_R^i(A_k, B)$.

(7) Let $G$ be a finite group. Let $P \to \mathbb{Z}$ denote a projective $\mathbb{Z}G$-resolution of the trivial module $\mathbb{Z}$, and $M$ a $\mathbb{Z}G$-module. Assume each $P_i$ is finitely generated.

   (a) Let $Q^n = \mathrm{Hom}_{\mathbb{Z}-mod}(P_n, \mathbb{Z})$ for $n \geq 0$. Show that $0 \to \mathbb{Z} \to Q^0 \to Q^1 \to \cdots$ is exact and $Q^n$ is $\mathbb{Z}G$-projective for all $n \geq 0$.

(b) By splicing $P \to \mathbb{Z}$ and $\mathbb{Z} \to Q$ obtain an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to Q^0 \to Q^1 \to Q^2 \to \cdots$$

of projective $\mathbb{Z}G$-modules. Set $F_i = P_i$ if $i \geq 0$ and $F_i = Q^{-i-1}$ if $i < 0$. The groups $\widehat{H}^n(G, M) = H^n(\mathrm{Hom}_{\mathbb{Z}G-mod}(F, M))$ are called the Tate cohomology groups of $G$. Prove that

$$\widehat{H}^n(G, M) \cong \begin{cases} H^n(G, M) & \text{if } n > 0 \\ \mathrm{Coker}\bar{N} & \text{if } n = 0 \\ \ker \bar{N} & \text{if } n = -1 \\ H_{-n-1}(G, M) & \text{if } n < -1 \end{cases}$$

where $\bar{N} : M_G \to M^G$ is as in (2).