MATH 9144A - ASSIGNMENT 1

(1) Categories

(a) Prove that if a group G is regarded as a category with one object then natural transformations on the identity functor are given by the central elements[1] in G.

(b) Assume $\mathbf{C}$ is complete. Prove that $\Delta \dashv \lim$ where Δ is the constant functor.

(c) Prove that $\mathrm{Hom}_{\mathbf{C}}(\coprod_I A_i, B) \cong \prod_I \mathrm{Hom}_{\mathbf{C}}(A_i, B)$.

(d) Prove that in an additive category the natural map $A \coprod B \to A \prod B$ is an isomorphism. If $T : \mathcal{A} \to \mathcal{B}$ is an additive functor between additive categories then prove that the natural map $T(A) \oplus T(B) \to T(A \oplus B)$ is an isomorphism.

(2) Exactness[2]

(a) Let $\mathbf{A}$ be an abelian category. Prove that $\mathrm{Hom}_{\mathbf{A}}(M, -)$ and $\mathrm{Hom}_{\mathbf{A}}(-, M)$ are left exact.

(b) Consider the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ in the category of abelian groups. Give an example of M such that $\mathrm{Hom}(M, -)$ fails to be right exact. Similarly find an N such that $\mathrm{Hom}(-, N)$ fails to be right exact.

(c) Prove that $M \otimes_R -$ is right exact on R-mod.

(d) In the category of abelian groups find an exact sequence and an M where $M \otimes -$ fails to be left exact.

(e) Give an example of a ring R and an R-module M such that $M \otimes_R -$ and $\mathrm{Hom}(M, -)$ are both exact.

(3) Cone & Cylinder

(a) Prove that the two obvious inclusions $(1, 0, 0) : A \to \mathrm{cyl}(\mathrm{Id}_A)$ and $(0, 0, 1) : A \to \mathrm{cyl}(\mathrm{Id}_A)$ are both chain homotopy equivalences.

(b) Prove that $f : A \to B$ is null homotopic $s : f \simeq 0$ if and only if f extends to $(-s, f) : \mathrm{cone}(\mathrm{Id}_A) \to B$.

(4) Group homology

(a) Let $G = \langle t | t^n = 1 \rangle$ and C denote the chain complex $\mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G$ (degree one and zero). Show that there is a long exact sequence

$$\cdots \to H_k(G) \to H_{k-2}(G) \to H_{k-1}(C) \to H_{k-1}(G) \to \cdots \to H_2(G) \to H_0(G) \to H_1(C) \to H_1(G) \to 0$$

[1] An element z is central if $gz = zg$ for all $g \in G$.

[2] See Weibel for the definition of left/right-exactness.

and deduce that $H_k(G) \cong H_{k-2}(G)$ for $k > 2$.
Hint: Start with a map $f : C \to P$ where P is the periodic free resolution of G constructed in class. Now identify the cokernel.

(b) Prove that $H_1(G) = G/[G, G]$ by using the bar resolution B.

(c) Let n denote the order of G. Prove that the map $H_k(G) \to H_k(G)$ given by multiplication by n is the zero map.
Hint: Consider the chain map $N : B \to B$ given by multiplication by the norm element $N = \sum_{g \in G} g$.