Homotopical approach to quantum contextuality

Cihan Okay

Bilkent University

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Quantum contextuality

Quantum contextuality is a fundamental feature of quantum mechanics first demonstrated in the theorems of Kochen–Specker[1] and Bell[2].

[1] Kochen and Specker, "The Problem of Hidden Variables in Quantum Mechanics".

[2] Bell, "On the Einstein Podolsky Rosen paradox".

Motivation

Contextuality has been established as a computational resource in two prominent schemes of quantum computation:

- measurement-based quantum computation[3] (MBQC),
- quantum computation with magic states[4] (QCM).

[3] Raussendorf, "Contextuality in measurement-based quantum computation".

[4] Howard et al., "Contextuality supplies the magic for quantum computation".

Linear constraint systems

A *linear constraint system*[5] is specified by a linear equation over $\mathbb{Z}/d$

$$\underbrace{\begin{bmatrix} M_{11} & \cdots & M_{1c} \\ \vdots & \ddots & \vdots \\ M_{r1} & \cdots & M_{rc} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_c \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_c \end{bmatrix}}_{b}$$

Instead of considering solutions over $\mathbb{Z}/d$ we will consider solutions over the unitary group $U(m)$.

[5] Cleve and Mittal, "Characterization of binary constraint system games".

Application

Sloftra's work[6] on Tsierelson's problem:

- wagon-wheel linear constraint system (over $\mathbb{Z}/2$) provides an embedding of any finitely presented group into a group associated to the linear constraint system.

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[6] Slofstra, "Tsirelson's problem and an embedding theorem for groups arising from non-local games".

Operators solutions

An *operator solution* over $U(m)$ consists of unitary matrices

$$\{A_i \in U(m) |\ 1 \leq i \leq c\}$$

satisfying the following conditions

- $(A_i)^d = I_m$ for all $1 \leq i \leq c$,
- $A_i A_j = A_j A_i$ whenever M_{ki} and M_{kj} are both $\neq 0$ for some $1 \leq k \leq r$,
- $A_1^{M_{k1}} A_2^{M_{k2}} \cdots A_c^{M_{kc}} = \omega^{b_k} I_m$ for all $1 \leq k \leq r$, where $\omega = e^{2\pi i/d}$.

Contextuality

An operator solution over $U(1)$ is called a *scalar solution*.

A linear constraint system is called *contextual* if it does not admit a scalar solution. Otherwise, it is called *non-contextual*.

Remark

Scalar solutions belong to μ_d*, the cyclic subgroup of* $U(1)$ *generated by* ω*.*

They can be identified with solutions of $Mx = b$ *over* $\mathbb{Z}/d$ *via the isomorphism* $\mu_d \to \mathbb{Z}/d$ *determined by* $\omega \mapsto 1$*.*

Hypergraph formulation

This data can be packaged as a pair $(\mathfrak{H}, \tau)$ where $\mathfrak{H} = (V, E, \epsilon)$ is a hypergraph

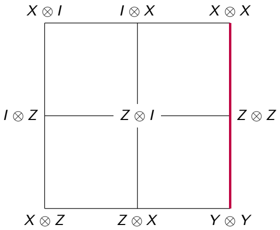
- vertex set $V = \{v_1, v_2, \cdots, v_c\}$
- edge set (contexts) $E = \{e_1, e_2, \cdots, e_r\}$ where $e_k = \{v_i | \ M_{ki} \neq 0\}$
- incidence weight $\epsilon_{e_k}(v_i) = M_{ki}$
- $\tau : E \to \mathbb{Z}/d$ is defined by $\tau(e_k) = b_k$.

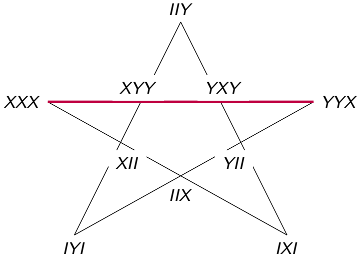
An operator solution will be regarded as a function

$$T : V \to U(m) \quad \text{where } T(v_i) = A_i.$$

Example - Mermin square and star

Contextual linear constraint systems defined over $\mathbb{Z}/2$

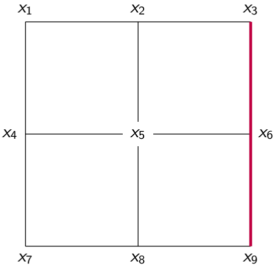




Each line specifies a context and τ takes the value 0 on each context except the colored ones.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Mermin square is contextual



$$
\begin{array}{r}
x_1 + x_2 + x_3 = 0 \\
x_4 + x_5 + x_6 = 0 \\
x_7 + x_8 + x_9 = 0 \\
x_1 + x_4 + x_7 = 0 \\
x_2 + x_5 + x_8 = 0 \\
\underline{x_3 + x_6 + x_9 = 1} \\
0 \neq 1
\end{array}
$$

Chain complex formulation

Chain complex associated to the hypergraph $\mathfrak{H} = (V, E, \epsilon)$:

$$C_*(\mathfrak{H}) : C_2 \xrightarrow{\partial} C_1 \xrightarrow{0} C_0$$

where $C_0 = \mathbb{Z}/d$, $C_1 = \mathbb{Z}/d[V]$, $C_2 = \mathbb{Z}/d[E]$

$$\partial[e] = \sum_{v \in e} \epsilon_e(v)\,[v]$$

There is a corresponding cochain complex $\mathcal{C}^*(\mathfrak{H})$

and $\tau : E \to \mathbb{Z}/d$ belongs to $C^2(\mathfrak{H}) = \{E \to \mathbb{Z}/d\}$.

Topological realization for hypergraphs

A *topological realization* for the hypergraph $\mathfrak{H}$ is a connected 2-dimensional cell complex $X(\mathfrak{H})$ with $X_1 = V$ and $X_2 = E$ together with a homomorphism of chain complexes

$$\begin{array}{ccccc} C_2(X) & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) \\ \| f_2 & & \| f_1 & & \downarrow f_0 \\ C_2(\mathfrak{H}) & \xrightarrow{\partial} & C_1(\mathfrak{H}) & \xrightarrow{0} & C_0(\mathfrak{H}) \end{array}$$

Typically $X(\mathfrak{H})$ has more than 1 vertices.

Homology and homotopy

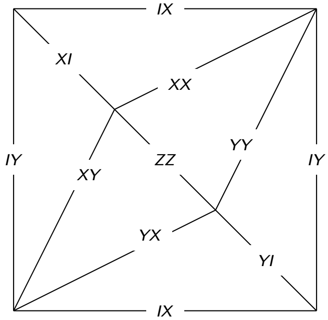
Theorem ([7])

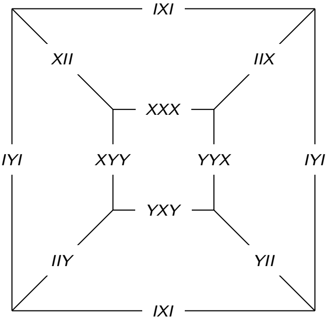
If $(\mathfrak{H}, \tau)$ *admits an operator solution but no scalar solutions then any topological realization* $X(\mathfrak{H})$ *has a non-trivial fundamental group i.e.* $\pi_1(X) \neq 1$.

$(\mathfrak{H}, \tau)$ admits a scalar solution if and only if $[\tau] = 0$ in $H^2(X(\mathfrak{H}))$ for any topological realization.

[7] Okay and Raussendorf, "Homotopical approach to quantum contextuality".

Example - topological realizations[8]





$[\tau] \neq 0$ in both of these examples.

[8] Okay et al., "Topological proofs of contextuality in quantum mechanics".

Next step

A linear constraint system can be turned into a pair consisting of a 2-dimensional cell complex and a 2-dimensional cohomology class

$$(\mathfrak{H}, \tau) \rightsquigarrow (X(\mathfrak{H}), [\tau])$$

Can we interpret operator solutions in a topological way?

Idea

Construct a "universal space" $\bar{B}_{d,m}$ such that any operator solution $T : V \to U(m)$ can be turned into a map

$$f_T : X(\mathfrak{H}) \to \bar{B}_{d,m}$$

defined up to homotopy.

This way we can associate

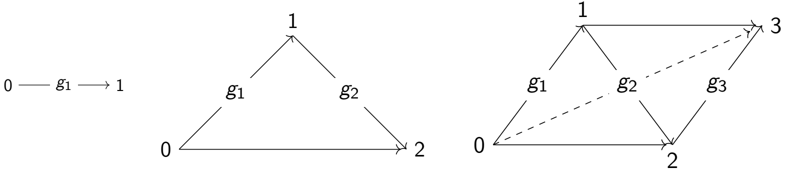
$$(\mathfrak{H}, \tau, T) \ \rightsquigarrow \ (X(\mathfrak{H}), [\tau], [f_T])$$

Classifying spaces

Let G be a group, e.g. the unitary group $U(m)$.

The classifying space BG is a cell complex constructed from n-tuples of group elements

$$(g_1, g_2, \cdots, g_n) \quad g_i \in G$$



Simplicial structure

More precisely, BG is the *geometric realization* of the simplicial space $\{G^n\}_{n\geq 0}$ whose simplicial structure maps are given by

$$d_i(g_1, g_2, \cdots, g_n) = \begin{cases} (g_2, g_3, \cdots, g_n) & i = 0 \\ (g_1, \cdots, g_i g_{i+1}, \cdots, g_n) & 0 < i < n \\ (g_1, g_2, \cdots, g_{n-1}) & i = n \end{cases}$$

$$s_j(g_1, g_2, \cdots, g_n) = (g_1, \cdots, g_j, 1, g_{j+1}, \cdots, g_n) \quad 0 \leq j \leq n$$

Commutative d-torsion version[9]

Let $B_{d,m} \subset BU(m)$ denote the cell complex consisting of n-tuples

$$(A_1, A_2, \cdots, A_n) \quad A_i \in U(m)$$

that satisfy

1. $(A_i)^d = I_m$ for all $1 \leq i \leq n$,
2. $A_i A_j = A_j A_i$ for all $1 \leq i, j \leq n$.

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[9] Adem, Cohen, and Torres Giese, "Commuting elements, simplicial spaces and filtrations of classifying spaces".

Classifying space for contextuality[10]

Let $\bar{B}_{d,m}$ denote the quotient space of $B_{d,m}$ obtained by

$$(A_1, \cdots, A_n) \sim (\alpha_1 A_1, \cdots, \alpha_n A_n) \quad \alpha_i \in \mu_d.$$

In fact, the quotient map

$$B_{d,m} \to \bar{B}_{d,m}$$

is a fibration sequence, i.e. homotopy groups are closely related,

with fiber $B\mu_d$.

It is classified by a cohomology class $\gamma_{d,m} \in H^2(\bar{B}_{d,m}, \mu_d)$.

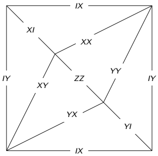
[10] Okay and Sheinbaum, "Classifying space for quantum contextuality".

Construction

Let $T : V \to U(m)$ be an operator solution of $(\mathfrak{H}, \tau)$

$$f_T : X \to \bar{B}_{d,m}$$

1. send each 0-cell to the unique 0-cell of $\bar{B}_{d,m}$
2. send the 1-cell labeled by $v \in V$ to the 1-cell $[T(v)]$
3. for each 2-cell $e \in E$ extend the map on the boundary to the interior of the disk.



Classification of operator solutions

We associate

$$(\mathfrak{H}, \tau, T) \rightsquigarrow (X(\mathfrak{H}), [\tau], [f_T]) \rightsquigarrow [f_T : X \to \bar{B}_{d,m}]$$

since $[\tau] = f_T^*(\gamma_{d,m})$.

Thus the emphasis has shifted from hypergraphs $\mathfrak{H}$ to 2-dimensional cell complexes X; from operators solutions to maps $X \to \bar{B}_{d,m}$.

Homotopy classes

Computing the homotopy classes of maps

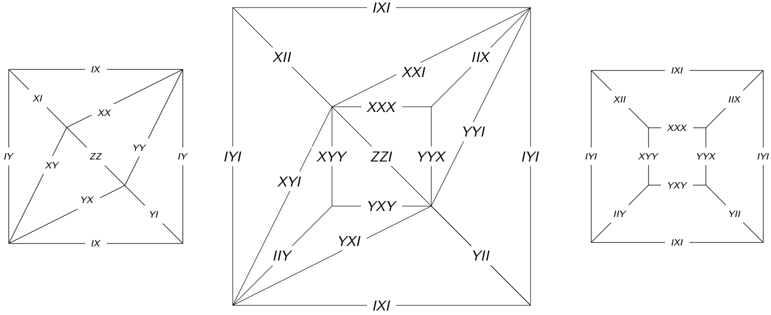
$$[X, \bar{B}_{d,m}] = \{f : X \to \bar{B}_{d,m}\}/\text{cont. def.}$$

provides a way to classify operator solutions.

This set can be computed algebraically:

Essentially it boils down to understanding π_1 and π_2 of $\bar{B}_{d,m}$.

Example - homotopic operator solutions



$T_1 = T_{sq} \otimes I_2$ and $T_2 = T_{st}$ give homotopic maps

$$[f_{T_1}] = [f_{T_2}] \in [S^1 \times S^1, \bar{B}_{2,2^3}]$$

Stabilization

Send A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and take the limit

$$U(1) \to U(2) \to \cdots \to U(m) \to U(m+1) \to \cdots \to U$$

By Bott periodicity

$$\pi_r(U) = \begin{cases} 0 & r \text{ even} \\ \mathbb{Z} & r \text{ odd} \end{cases}$$

Complex K-theory

This is a generalized cohomology theory used in the stable classification of vector bundles.

Complex K-theory is defined as the set of homotopy classes

$$\tilde{K}(X) = [X, BU]$$

where

$$BU(1) \to BU(2) \to \cdots \to BU(m) \to BU(m+1) \to \cdots \to BU.$$

Commutative d-torsion K-theory[11]

This is a generalized cohomology theory defined by

$$k\mu_d(X) = [X, B_d^{\mathbb{S}}]$$

where

$$B_{d,1} \to B_{d,2} \to \cdots \to B_{d,m} \to B_{d,m+1} \to \cdots \to B_d^{\mathbb{S}}$$

It is used in the stable classification of vector bundles with "commutative d-torsion structure" on their transition functions.

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[11] Adem et al., "Infinite loop spaces and nilpotent K–theory"; Gritschacher, "Commutative K-theory".

Stable classifying space for contextuality

The stabilization process $B_{d,m} \rightsquigarrow B_d^{\mathbb{S}}$ cannot be used for $\bar{B}_{d,m}$.

$$\omega^k I_{m+1} \neq \begin{pmatrix} \omega^k I_m & 0 \\ 0 & 1 \end{pmatrix}$$

i.e. the diagram does not commute:

$$\begin{array}{ccc} B\mu_d & = & B\mu_d \\ \downarrow & & \downarrow \\ B_{d,m} & \longrightarrow & B_{d,m+1} \end{array}$$

$C(d,m)$-cohomology

However, we can let $C(d,m)$ to be a "quotient" of $k\mu_d$ in the *stable homotopy category* and let $\bar{B}^{\mathbb{S}}_{d,m}$ to be the space representing this cohomology theory

$$C(d,m)(X) = [X, \bar{B}^{\mathbb{S}}_{d,m}]$$

More precisely, we consider the cofiber sequence

$$\mathbb{S} \wedge B\mu_d \xrightarrow{\delta_m} k\mu_d \to C(d,m)$$

where these maps are defined as maps of Γ-spaces.

Stable classification

This construction comes with a map $\bar{\iota} : \bar{B}_{d,m} \to \bar{B}^{\mathbb{S}}_{d,m}$

$$(\mathfrak{H}, \tau, T) \;\rightsquigarrow\; [f_T : X \to \bar{B}_{d,m}] \;\rightsquigarrow\; [\hat{f}_T : X \to \bar{B}^{\mathbb{S}}_{d,m}]$$

where $\hat{f}_T = \bar{\iota} f_T$.

Therefore elements of $C(d,m)(X)$ label "stable classes" of operator solutions.

$C(d, m)$-cohomology of a space

Theorem (O.)

There is an isomorphism

$$C(d, m)(X) \cong H^1(X, \pi_1) \oplus H^2(X, \pi_2)$$

where

$$0 \to \pi_2 \to \mathbb{Z}/d \xrightarrow{\times m} \mathbb{Z}/d \to \pi_1 \to 0.$$

π_1 and π_2 of $\bar{B}^{\mathbb{S}}_{d,m}$ are both isomorphic to $\mathbb{Z}/\gcd(d, m)$.

Stable contextuality

Corollary

If d and m are coprime then $C(d,m)(X) = 0$ for any X.

In particular, any $(\mathfrak{H}, \tau)$ over $\mathbb{Z}/d$ admits a scalar solution if it admits a solution over $U(m)$.

We can associate

$$(\mathfrak{H}, \tau, T) \rightsquigarrow [f_T : X \to \bar{B}_{d,m}] \rightsquigarrow [\hat{f}_T : X \to \bar{B}^{\mathbb{S}}_{d,m}] = (\varphi_1; \varphi_2)$$

where $\varphi_i \in H^i(X, \mathbb{Z}/\gcd(d,m))$ and

$$H^2(X, \mathbb{Z}/\gcd(d,m)) \to H^2(X, \mathbb{Z}/d)$$

$$\varphi_2 \mapsto [\tau]$$

Mermin class

Let T_n be the operator solution $T_{sq} \otimes I_{2^{n-1}}$ over $U(2^n)$ for the Mermin square.

The map $\hat{f}_{T_n}$ factors as

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\hat{f}_{T_n}} & \bar{B}^{\mathbb{S}}_{2,2^n} \\ \downarrow & \nearrow_{M_n} & \\ S^2 & & \end{array}$$

where $[M_n]$ corresponds to the non-trivial element in

$$C(2, 2^n)(S^2) \cong H^2(S^2, \mathbb{Z}/2) = \mathbb{Z}/2.$$

Inverse problem

For simplicity let $m = d^n$.

Thus $\mathbb{Z}/\gcd(d,m) = \mathbb{Z}/d$ and

$$C(d,d^n)(X) = H^1(X) \oplus H^2(X)$$

$$(\varphi_1;\varphi_2) = [\hat{f}_T : X \to \bar{B}^{\mathbb{S}}_{d,m}] \overset{?}{\rightsquigarrow} [f_T : X \to \bar{B}_{d,m}] \overset{?}{\rightsquigarrow} (\mathfrak{H}, \tau, T)$$

Relation to symmetry-protected topological phases[12]

lattice models $\rightsquigarrow$ generalized cohomology, e.g. K-theory

operator solutions $\rightsquigarrow$ $C(d,m)$-cohomology obtained from $k\mu_d$

Real version:

$$0 \to \underbrace{\pi_2 ko_{\mathrm{sym}}}_{[GW]\ \mathrm{SPT}} \to \pi_2 C_{\mathbb{R}}(2, 2^n) \to \underbrace{H^2(S^2, \mathbb{Z}/2)}_{[M_n]\ \mathrm{LCS}} \to 0$$

[12] Kitaev, *On the classification of short-range entangled states*.

Thank you for your attention!