On Quillen's conjecture

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Abstract: Quillen’s conjecture relates an algebraic invariant and a homotopy invariant of a finite group. The conjecture is known to hold for several families of groups since the work of Quillen, Aschbacher, Smith and Alperin in the 80’s and 90’s. Here we present a new geometric approach to the subject.

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G finite group, p is a prime

On the algebraic side:

Two posets:

$$\mathcal{S}_p(G) = \{ 1 \neq \overbrace{P}^{p\text{-subgroup}} \leq G \} \quad (\text{inclusion})$$
$$\cup\!\!\!\mid$$
$$\mathcal{A}_p(G) = \{ 1 \neq \underbrace{E}_{\text{elementary abelian } p\text{-subgroup}} \leq G \} \quad (\text{inclusion})$$

$\mathcal{P}$ poset $\Rightarrow |\mathcal{P}|$ top. space

simplicial complex where n-simplices are n-chains in the poset: $\underbrace{P_0 < P_1 < \dots < P_n}_{n\text{-simplex}}$

$O_p(G)$ = largest p-subgroup normal in G.
$_p\trianglelefteq G$

Theorem: G finite group, p prime. $O_p(G) \neq 1 \Rightarrow |\mathcal{S}_p(G)| \simeq *$.

Proof: (sketch).

$\mathcal{P} \overset{f}{\underset{g}{\rightrightarrows}} \mathcal{Q}$ f, g maps of posets (they preserve order).

If $f \leq g \Rightarrow |\mathcal{P}| \overset{|f|}{\underset{|g|}{\rightrightarrows}} |\mathcal{Q}|$ $|f| \simeq |g|$.

$$\mathcal{S}_p(G) \overset{id}{\underset{c}{\overset{r}{\rightrightarrows}}} \mathcal{S}_p(G)$$

Conic contraction

$$\left.\begin{array}{l} H \to H = id(H) \\ \quad\ \ \rotatebox{90}{$\leq$} \\ H \to H\,O_p(G) = r(H) \\ \quad\ \ \rotatebox{90}{$\geq$} \\ H \to O_p(G) = c(H) \end{array}\right] \Rightarrow |Id| \simeq |c|$$

$\Downarrow$

$|\mathcal{S}_p(G)| \simeq *$.

Quillen's conjecture: $\boxed{|\mathcal{S}_p(G)| \simeq * \Rightarrow O_p(G) \neq 1.}$

Quillen's conjecture: $|S_p(G)| \simeq * \implies O_p(G) \neq 1$.

known results: $r = \mathrm{rank}_p(G)$ = dimension of the largest elem. ab. p-subgroup of G.

$\boxed{|S_p(G)| \simeq |A_p(G)|}$ Quillen

$(C_p, \; C_p \times C_p, \; C_p \times C_p \times C_p, \ldots)$
$r=1 \quad r=2 \quad r=3$

- $r = 0,1$ trivial
- $r=2$: Quillen proved this case.

(1978) $|S_p(G)| \simeq |A_p(G)| \leftarrow$ 1-dimensional complex $\Big] \implies |A_p(G)|$ tree $\circlearrowleft G$

$|A_p(G)| \simeq *$

$\xrightarrow{\text{Serre}} \exists$ fix point. $\implies O_p(G) \neq 1$

- $r=3$ (2019) Piterman, Sadofschi, Viruel

- G solvable Quillen '78

G p-solvable Alperin (coprime action results, CFSG)

- G doesn't have unitary component $\underline{U_n(q)}$ $q \equiv -1 \pmod p$, $p > 5$

Aschbacher–Smith '93 (CFSG)

Classification of Finite Simple Groups: (CFSG).

- Alternating
- Lie type in non-defining characteristic $(GL_n(q), p \nmid q)$
- Lie type in defining char. $(GL_n(q), q = p^m)$

Quillen '78

- Sporadic

$\boxed{|S_p(G)|} \simeq$ Tits building $\simeq \bigvee S^{\ell}$

$\ell = \mathrm{rank}_{Lie}(G)$ fixed

$q = p^m$ $\quad \mathrm{rank}_p(G) \xrightarrow[m \to \infty]{} +\infty$

What is expected in non-defining characteristic?

$\boxed{\begin{array}{l} (QD_p) \text{ (Quillen dimension property):} \\ G \text{ has } QD_p \text{ if } \tilde{H}_{r-1}(|S_p(G)|) \neq 0 \end{array}}$ $(r = \mathrm{rank}_p(G))$.

$(\dim |A_p(G)| = r-1)$.

Geometric approach to prove QD_p property for families of finite simple groups.

G, p prime, $r = \mathrm{rank}_p(G)$, $E = \langle e_1, \ldots, e_r \rangle$, $O_p(G) = 1$.

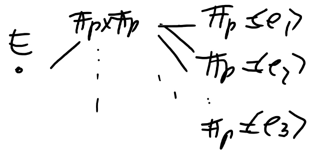
¿ $\tilde{H}_{r-1}(|A_p(G)|) \neq 0$?

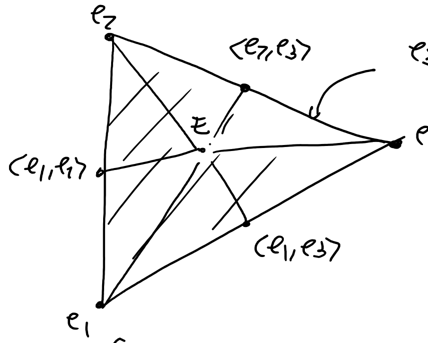
$E \leq G \rightsquigarrow |A_p(E)| \subseteq |A_p(G)|$

$E \leq G \rightsquigarrow |\mathcal{A}_p(E)| \subseteq |\mathcal{A}_p(G)|$

Example: $E = \langle e_1, e_2, e_3 \rangle$ $(r=3)$

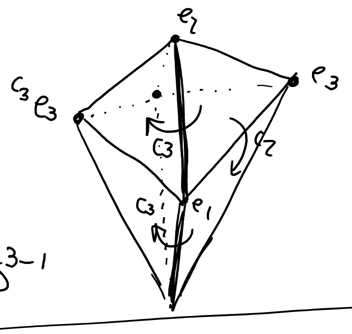
$\mathcal{A}_p(E) =$





Bar. subdiv of Δ^2

$G \curvearrowright E$ $\quad$ $\curvearrowright G$ $|\mathcal{A}_p(E)|$



$c_3 \in C_G(\langle e_1, e_2 \rangle) \setminus N_G(e_3)$

$c_1 \in C_G(\langle e_2, e_3 \rangle) \setminus N_G(e_1)$

$c_2 \in C_G(\langle e_1, e_3 \rangle) \setminus N_G(e_2)$

$[c_1, c_2] = 1$

$[c_2, c_3] = 1$

octahedron $\simeq S^2 = S^{3-1}$

Theorem: G finite group, p prime, $r = \mathrm{rank}_p(G)$, $E = \langle e_1, \ldots, e_r \rangle$
$c_i \in C_G(\langle e_1, \ldots, \hat{e}_i, \ldots, e_r \rangle) \setminus N_G(E)$, $[c_i, c_j] = 1$ $\forall i,j = 1, \ldots, r$
Then $\tilde{H}_{r-1}(|\mathcal{A}_p(G)|) \neq 0$

- solvable ✓
- p-solvable CFSG to find c_i's ✓.

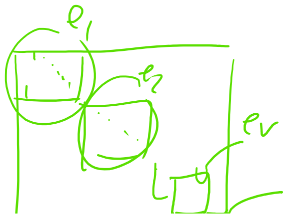
Lie type groups in non-def. char and alternating/symmetric groups.

Symmetric group:

$G = \Sigma_n$, p, $r = \lfloor \frac{n}{p} \rfloor$ ¿e_i? ¿c_i?

$e_1 = (1\ 2 \cdots p)$

$e_2 = (p+1 \cdots 2p)$



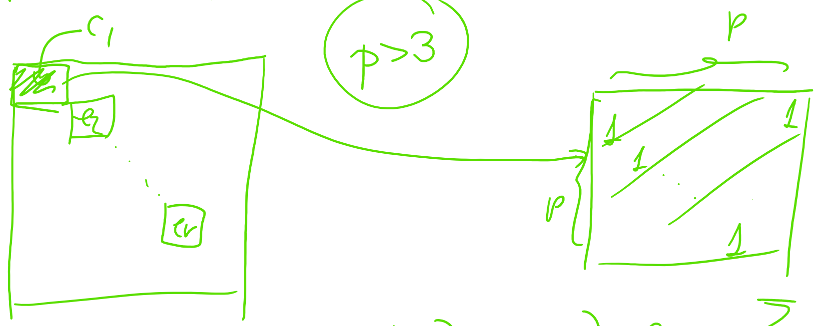
$c_1 \in C_G(\underbrace{e_2, \ldots, e_r}) \setminus N_G(E)$

$e_2 = (p+1 \cdots 2p)$

$\vdots$

$e_r \ ((r-1)p+1, \dots, rp)$

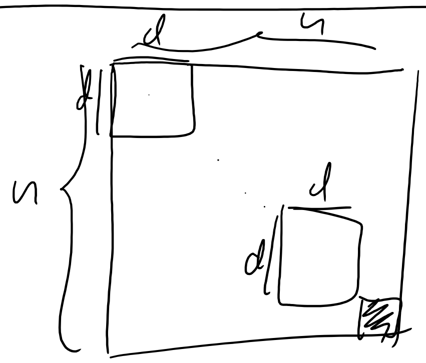
$C_1 \in C_G(e_1, \dots, e_r)$



$p>3 \Rightarrow \exists\, c_1 \in \Sigma_p \setminus N_G(e_1)$

$GL_n(q)$, p prime, d = smallest integer s.t. $q^d - 1 \equiv 0 \pmod p$.

$r = \left\lfloor \frac{n}{d} \right\rfloor$

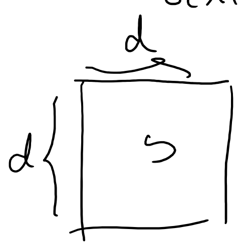


int $q^d - 1$

$GL_1(q^d) \longrightarrow GL_d(q) \xrightarrow{e_i} GL_n(q)$

$x \longrightarrow y \longrightarrow \begin{pmatrix} I & & \\ & s & \\ & & I \end{pmatrix} = e_i$

$o(x) = p \mid (q^d - 1)$



$d>1 \Rightarrow \exists\, c_i \in GL_d(q) \setminus N_G(s).$

$SL_n(q)$, $d=1$: $\left((n,p) = 1 \right)$.

$r = \text{rank} = n-1$ $\quad u \in GL_1(q) \mid o(u) = p \mid q-1$

$e_1 = \begin{pmatrix} u^{1-n} & & & \\ & u & & \\ & & \ddots & \\ & & & u \end{pmatrix}$ $\quad e_2 = \begin{pmatrix} u & & & & \\ & u^{1-n} & & & \\ & & u & & \\ & & & \ddots & \\ & & & & u \end{pmatrix}$ $\quad \cdots \quad$ $e_{n-1} = \begin{pmatrix} u & & & \\ & \ddots & & \\ & & u^{1-n} & \\ & & & u \end{pmatrix}$

$E = \langle e_1, \dots, e_n \rangle$

¿$c_1, \dots, c_{n-1}$?

¿ $c_1, \dots, c_{n-1}$?

$$c_1 = \begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & \ddots & \\ -1 & & & 1 \end{pmatrix} \qquad c_2 = \begin{pmatrix} 1 & & & \\ & \ddots & & 1 \\ -1 & & & \\ & & & 1 \end{pmatrix} \qquad \cdots \qquad c_{n-1} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & & 1 \\ & & -1 & 1 \end{pmatrix}$$

$[c_i, c_j] = 1$ $\qquad c_i \in C_G(\langle e_1, \dots, \hat{e}_i, \dots, e_{n-1} \rangle) \setminus N_G(E)$.

¿ $H_n(G)$?

$c_1 c_2 = c_2 c_1 \longrightarrow$ abelian group $\to$ sphere Z^n

$c_1 c_2 c_1 = c_2 c_1 c_2 \longrightarrow$ braid group $\to$ permutahedron $n!$