QUANTUM CHANNELS

We will write $T(V,W)$ for the set of linear maps

$$\Phi : L(V) \longrightarrow L(W).$$

The set $T(V,W)$ is a vector space:

1) $\Phi + \Psi$ is defined by

$$(\Phi + \Psi)(A) = \Phi(A) + \Psi(A).$$

2) $\alpha \Phi$ for $\alpha \in \mathbb{C}$ is defined by

$$(\alpha \Phi)(A) = \alpha \Phi(A).$$

The adjoint of Φ is the linear map $\Phi^{+} : L(W) \longrightarrow L(V)$ uniquely specified by the equation

$$\langle \Phi^{+}(Y), X \rangle = \langle Y, X \rangle.$$

Given $\Phi \in T(V_1, W_1)$ and $\Psi \in T(V_2, W_2)$ we can define the tensor product:

$$\Phi \otimes \Psi \in T(V_1 \otimes V_2, W_1 \otimes W_2)$$

to be the unique linear map satisfying

$$(\Phi \otimes \Psi)(A \otimes B) = \Phi(A) \otimes \Psi(B).$$

Ex: Partial trace $\mathrm{Tr}_W \in T(V \otimes W, V)$ is defined by

$$\mathrm{Tr}_W(A \otimes B) = \mathbb{1}_{L(V)} \otimes \mathrm{Tr}\,(A \otimes B)$$
$$= A \otimes \mathrm{Tr}(B)$$
$$= \mathrm{Tr}(B)\, A$$

The adjoint $\mathrm{Tr}_W^{+} : L(V) \to L(V \otimes W)$ is given by

$$\mathrm{Tr}_W^{+}(A) = \sum_{a,b} \langle |ab\rangle\langle ab| , \mathrm{Tr}_W^{+}(A) \rangle \, |ab\rangle\langle ab|$$
$$= \sum_{a,b} \overline{\langle \mathrm{Tr}_W^{+}(A) , |ab\rangle\langle ab| \rangle} \, |ab\rangle\langle ab| \quad (\mathrm{Tr}_W)$$
$$= \sum_{a,b} \langle A , |a\rangle\langle a| \rangle \, |ab\rangle\langle ab|$$
$$= \underbrace{\sum_{a} \langle a, Aa\rangle \, |a\rangle\langle a|}_{A} \otimes \underbrace{\sum_{b} |b\rangle\langle b|}_{\mathbb{1}_W}$$
$$= A \otimes \mathbb{1}_W$$

A linear map $\Phi : L(V) \to L(W)$ is

1) positive if

$$\Phi(A) \in \mathrm{Pos}(W) \quad \forall A \in \mathrm{Pos}(V),$$

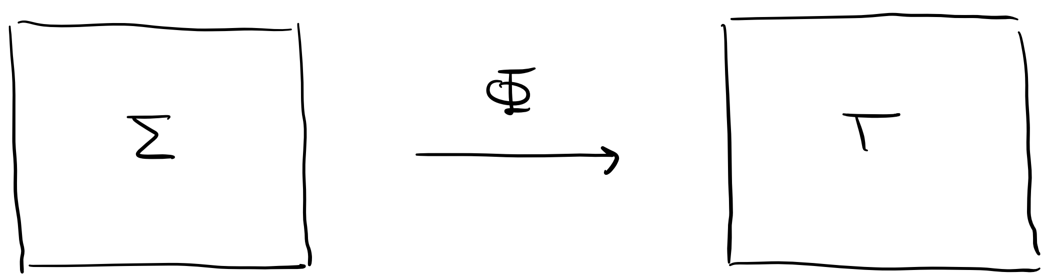
2) completely positive if $\Phi \otimes \mathbb{1}_{L(U)}$ is positive $\forall\, U$,

3) trace preserving if

$$\mathrm{Tr}\, \Phi(A) = \mathrm{Tr}(A) \quad \forall A \in L(V),$$

4) unital if $\Phi(\mathbb{1}_V) = \mathbb{1}_W$.

A quantum channel is a completely positive trace preserving linear map $\Phi : L(V) \to L(W)$. We will write $C(V, W)$ for the set of quantum channels and write simply $C(V)$ for $C(V, V)$.



$\rho \in \mathrm{Den}(\mathbb{C}\Sigma)$ $\qquad$ $\Phi(\rho) \in \mathrm{Den}(\mathbb{C}\Gamma)$

Ex:

a) Isometric channels:

For $U \in \mathrm{U}(Y, W)$ we can define

$$\Phi(A) = U A U^{\dagger}.$$

We have

1) $\Phi \otimes \mathbb{1}_{L(U)}(P) = (U \otimes \mathbb{1}_U) P (U^{\dagger} \otimes \mathbb{1}_U)$ is positive $\forall P \in \mathrm{Pos}(V \otimes U)$;

2) $\mathrm{Tr}(U A U^{\dagger}) = \mathrm{Tr}(A)$.

Therefore $\Phi \in \mathrm{C}(Y, W)$.

b) Replacement channels:

For $\sigma \in \mathrm{Den}(W)$ we can define

$$\Phi(A) = \mathrm{Tr}(A)\, \sigma.$$

We have

1) $\Phi \otimes \mathbb{1}_{L(U)}(B) = \sigma \otimes \mathrm{Tr}_V(B)$, which is positive if B is positive.

2) $\mathrm{Tr}(\mathrm{Tr}(A)\, \sigma) = \mathrm{Tr}(A)\, \mathrm{Tr}(\sigma) = \mathrm{Tr}(A)$.

When $\sigma = \mathbb{1}_V / \dim V$ this channel is called the completely depolarizing channel and is denoted by

$$\Omega(A) = \mathrm{Tr}(A) \frac{\mathbb{1}_V}{\dim V}.$$

c) Completely dephasing channel

This is a linear map

$$\Delta : L(\mathbb{C}\Sigma) \to L(\mathbb{C}\Sigma)$$

defined by

$$\Delta(A) = \sum_{a \in \Sigma} A(a,a) \, |a\rangle\langle a| .$$

<u>Classical channels</u>

A stochastic operator is a linear map $S : \mathbb{R}\Sigma \to \mathbb{R}\Gamma$ such that

1) $S(a,b) \geq 0 \quad \forall a, b,$

2) $\sum_{a \in \Sigma} S(a,b) = 1 \quad \forall b .$

We have

$$\sum_a Sp(a) = \sum_a \sum_b S(a,b) \, p(b)$$

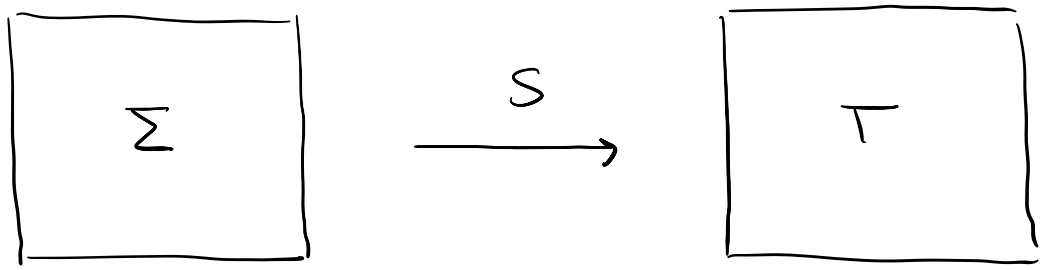
$$= \sum_b \underbrace{\sum_a S(a,b)}_{1} \, p(b)$$

$$= \sum_b p(b) = 1 .$$

Therefore

$$S(p) \in \text{Dist}(\Gamma), \quad \forall p \in \text{Dist}(\Sigma) .$$

A classical channel is represented by a stochastic operator:



$p \in \mathrm{Dist}(\Sigma)$ $\qquad\qquad$ $S(p) \in \mathrm{Den}(\mathbb{C}\Gamma)$

Associated to S we can define a quantum channel

$$\Phi_S(A) = \sum_b \Big(\sum_a S(b,a)\, A(a,a) \Big) |b\rangle\langle b|$$

When $S = \mathbb{1}$ we have

$$\Phi_S = \Delta .$$

Exercise: Verify that this is indeed a channel. You can use the characterization that will show up in later sections.

Pro: If $\Phi \in T(V,W)$ is positive then Φ^+ is also positive.

Proof: For $Q \in Pos(W)$ and $P \in Pos(V)$ we have

$$\langle \Phi^+(Q), P \rangle = \langle Q, \Phi(P) \rangle \geq 0.$$

Therefore $\Phi^+(Q)$ is positive. ☒

Cor: $Tr : L(V) \to \mathbb{C}$ is a channel

Proof: To show that $Tr \otimes \mathbb{1}_{L(U)}$ is positive it suffices to show that $Tr^+ \otimes \mathbb{1}_{L(U)}$ is positive. For $P \in Pos(U)$ we have

$$Tr^+ \otimes \mathbb{1}_{L(U)}(P) = \mathbb{1}_V \otimes P,$$

which is positive.

We also have $Tr(Tr(A)) = Tr(A)$. ☒

Pro: The tensor product $\Phi \otimes \Psi$ of two channels is a channel.

(channels of this form are called product channels)

Proof: Complete positivity follows from the fact that tensor product of positive operators is positive. Trace preserving property follows from $Tr(A \otimes B) = Tr(A)\,Tr(B)$. ☒

Cor: $Tr_W = \mathbb{1}_{L(V)} \otimes Tr \in C(V \otimes W, V)$.

Representations of channels

Let $\Phi : L(V) \to L(W)$ be a channel.

1) Natural representation:

This is a bijective linear map

$$K : T(V, W) \longrightarrow L(V \otimes V, W \otimes W)$$

defined by

$$K(\Phi)\big(\mathrm{vec}(A)\big) = \mathrm{vec}(\Phi(A)).$$

Pro: $K(\Phi^{+}) = (K(\Phi))^{+}$.

Proof: We have

$$\Phi\big(|a\rangle\langle b|\big) = \sum_{c,d} \Phi(ab, cd)\; |c\rangle\langle d|.$$

and

$$K(\Phi)\,|ab\rangle = \sum_{c,d} \Phi(ab, cd)\; |cd\rangle. \qquad \square$$

2) Choi-Jamiolkowski representation:

This is a bijective linear map

$$J : T(V, W) \longrightarrow L(W \otimes V)$$

defined by

$$J(\Phi) = \Phi \otimes \mathbb{1}_{L(V)}\big(\mathrm{vec}\,\mathbb{1}_V\,(\mathrm{vec}\,\mathbb{1}_V)^{+}\big).$$

$$= \Phi \otimes \mathbb{1}_{L(V)}\Big(\sum_{a,b} |aa\rangle\langle bb|\Big)$$

$$= \sum_{a,b} \Phi\big(|a\rangle\langle b|\big) \otimes |a\rangle\langle b|.$$

The rank of $J(\Phi)$ is called the Choi rank.

Pro: $\Phi(A) = \mathrm{Tr}_V(J(\Phi)(\mathbb{1}_W \otimes A^T))$.

Proof: We have

$$\mathrm{Tr}_V(J(\Phi)(\mathbb{1}_W \otimes A^T))$$

$$= \mathrm{Tr}_V\Big(\sum_{a,b} \Phi(|a\rangle\langle b|) \otimes |a\rangle\langle b| A^T\Big)$$

$$= \sum_{a,b} \Phi(|a\rangle\langle b|) \otimes \mathrm{Tr}(|a\rangle\langle b| A^T)$$

$A^T = \sum_{c,d} A(d,c)\, |c\rangle\langle d|$

$$= \sum_{c,d} A(d,c)\, \Phi(|d\rangle\langle c|)$$

$$= \Phi(A). \qquad \square$$

This gives the inverse map

$$J^{-1}: L(W \otimes V) \longrightarrow T(V,W) \quad \text{(linear)}$$

defined by $\quad J^{-1}(C) = \mathrm{Tr}_V(C(\mathbb{1}_W \otimes (-)^T))$.

3) Kraus representation:

For $\{A_a, B_a \in L(V,W)\}_{a\in\Sigma}$ we define

$$\Phi : L(V) \to L(W)$$

by the formula

$$\Phi(A) = \sum_a A_a A B_a^{\dagger}.$$

This representation is not unique.

Pro: $\Phi^{\dagger}(A) = \sum_a A_a^{\dagger} A B_a$.

Proof: Follows from cyclicity of trace. $\square$

4) Stinespring representation:

For $A, B \in L(V, W \otimes U)$ we define

$$\Phi(C) = \mathrm{Tr}_U(ACB^+).$$

This representation is also not unique.

<u>Pro</u>: $\Phi^+(C) = A^+(C \otimes \mathbb{1}_U)B$.

<u>Proof</u>: We have

$$\langle D, \Phi(C)\rangle = \langle D, \mathrm{Tr}_U(ACB^+)\rangle$$
$$= \langle D \otimes \mathbb{1}_U, ACB^+\rangle$$
$$= \mathrm{Tr}(D^+ \otimes \mathbb{1}_U ACB^+)$$
$$= \langle \underbrace{A^+ D \otimes \mathbb{1}_U B}_{\Phi^+(D)}, C\rangle$$

☒

<u>Pro</u>: The following are equivalent.

1) $K(\Phi) = \sum_a A_a \otimes \overline{B}_a$.

2) $J(\Phi) = \sum_a \mathrm{vec}\, A_a (\mathrm{vec}\, B_a)^+$.

3) $\Phi(A) = \sum_a A_a A B_a^+$

4) For $U = \mathbb{C}\Sigma$,

$A = \sum_a A_a \otimes |a\rangle$ and $B = \sum_a B_a \otimes |a\rangle$

we have $\Phi(C) = \mathrm{Tr}_U(ACB^+)$.

Proof: $(3 \Rightarrow 1)$: We have

$$K(\Phi)\, \mathrm{vec}(A) = \mathrm{vec}(\Phi(A))$$
$$= \sum_a \mathrm{vec}(A_a A B_a^\dagger)$$
$$\overset{\text{Lem}}{=} \sum_a (A_a \otimes \overline{B}_a)(\mathrm{vec}\, A)$$

$(3 \Rightarrow 2)$: We have

$$J(\Phi) = \Phi \otimes \mathbb{1}_{L(V)} (\mathrm{vec}\, \mathbb{1}_V (\mathrm{vec}\, \mathbb{1}_V)^\dagger)$$
$$= \sum_a \underbrace{A_a \otimes \mathbb{1}_V (\mathrm{vec}\, \mathbb{1}_V)}_{\mathrm{vec}\, A_a} \underbrace{(\mathrm{vec}\, \mathbb{1}_V)^\dagger B_a^\dagger \otimes \mathbb{1}_V}_{(\mathrm{vec}\, B_a)^\dagger} \quad \text{(Lem)}$$
$$= \sum_a \mathrm{vec}\, A_a (\mathrm{vec}\, B_a)^\dagger .$$

$(1, 2 \Rightarrow 3)$: Similar. (exercize)

$(4 \Rightarrow 3)$: We have

$$\mathrm{Tr}_U (A C B^\dagger) = \sum_{a,b} \mathrm{Tr}_U (A_a \otimes |a\rangle\, C\, B_b^\dagger \otimes \langle b|)$$
$$= \sum_a A_a C B_a^\dagger$$
$$= \Phi(C).$$

$(3 \Rightarrow 4)$: Similar. (exercize) ☐

Cor: Let $\Phi \in T(V, W)$ and $r = \mathrm{rank}(J(\Phi))$.

1) There exists a Kraus representation with $|\Sigma| = r$.

2) There exists a Stinespring representation with $U = \mathbb{C}^\Sigma$.

Proof: Let $\{u_a : a \in \Sigma\}$ be an orthonormal basis for $\mathrm{im}(J(\Phi))$. Since $\mathrm{rank}\, J(\Phi) = r$ we have $|\Sigma| = r$.

Then we can write

$$J(\Phi) = \sum_a |u_a\rangle\langle v_a|$$

where

$$|v_a\rangle = J(\Phi)\,|u_a\rangle.$$

Let A_a and B_a be such that

$$\mathrm{vec}\, A_a = |u_a\rangle \quad \text{and} \quad \mathrm{vec}\, B_a = |v_a\rangle.$$

Then

$$J(\Phi) = \sum_a \mathrm{vec}\, A_a \,(\mathrm{vec}\, B_a)^{\dagger}.$$

Then the Proposition gives the Kraus and Stinespring representation. ☒

Given $|u_a\rangle$ the vectors $|v_a\rangle$ are uniquely determined:

$$J(\Phi) = \sum_{a,b} \alpha_{ab}\, |u_a\rangle\langle b|$$

since $\{|u_a\rangle\}$ is a basis for $\mathrm{im}(J(\Phi))$

$$= \sum_a |u_a\rangle \Big(\underbrace{\sum_b \overline{\alpha}_{ab}\, |b\rangle}_{|v_a\rangle} \Big)^{\dagger}$$

$$= \sum_a |u_a\rangle\langle v_a|.$$

Characterizations of completely positive maps

For $\Phi \in T(V, W)$ the following are equivalent. (Φ nonzero)

1) Φ is completely positive
2) $\Phi \otimes \mathbb{1}_{L(V)}$ is positive
3) $J(\Phi) \in \mathrm{Pos}(W \otimes V)$.
4) There exist $\{A_a \in L(V,W)\}_{a \in \Sigma}$ where $|\Sigma| = r$ such that
$$\Phi(C) = \sum_a A_a C A_a^\dagger .$$
5) There exist $A \in L(V, W \otimes \mathbb{C}^\Sigma)$ where $|\Sigma| = r$ such that
$$\Phi(C) = \mathrm{Tr}_{\mathbb{C}^\Sigma}(A C A^\dagger).$$

Proof: $(1 \Rightarrow 2)$: By definition.

$(2 \Rightarrow 3)$: This follows from
$$\langle v, \mathrm{vec}\, \mathbb{1}_V (\mathrm{vec}\, \mathbb{1}_V)^\dagger v \rangle = \langle (\mathrm{vec}\, \mathbb{1}_V)^\dagger v, (\mathrm{vec}\, \mathbb{1}_V)^\dagger v \rangle \geq 0,$$
i.e. $\mathrm{vec}\, \mathbb{1}_V (\mathrm{vec}\, \mathbb{1}_V)^\dagger$ being positive.

$(3 \Rightarrow 4)$: By spectral decomposition:
$$J(\Phi) = \sum_a \lambda_a |v_a\rangle\langle v_a|, \quad \lambda_a \in \mathbb{R}_{\geq 0}$$
$$= \sum_a \sqrt{\lambda_a} |v_a\rangle\langle v_a| \sqrt{\lambda_a}$$
$$= \sum_a \mathrm{vec}\, A_a (\mathrm{vec}\, A_a)^\dagger$$
where $\mathrm{vec}\, A_a = \sqrt{\lambda_a} |v_a\rangle$.

Then the Proposition gives the Kraus representation.

(4 ⇒ 1): Given $P \in \mathrm{Pos}(V \otimes U)$ the operator

$$\Phi \otimes \mathbb{1}_{L(U)}(P) = \sum_a \underbrace{(A_a \otimes \mathbb{1}_U) P (A_a^\dagger \otimes \mathbb{1}_U)}$$

This is positive: Write $P = BB^\dagger$ then

$$A_a \otimes \mathbb{1}_U \, P \, A_a^\dagger \otimes \mathbb{1} = CC^\dagger \quad \text{where}$$

$$C = A_a \otimes \mathbb{1}_U B.$$

is positive since sum of positive operators is positive. (Exercise).

(4 ⇒ 5): By the Proposition.

(5 ⇒ 1): For positive P we have

$$\Phi(C) \otimes \mathbb{1}_{L(U)}(P) = \mathrm{Tr}_{\mathbb{C}\Sigma}(\underbrace{A \otimes \mathbb{1}_U \, P \, A^\dagger \otimes \mathbb{1}_U}_{\text{positive}})$$

by positivity of the partial trace. ◻

<u>Unitary equivalence of Krauss representations</u>

Let $\Phi(A) = \sum_a A_a A A_a^\dagger$ and

$\Psi(A) = \sum_a B_a A B_a^\dagger$ be such that

$$\Phi(A) = \Psi(A) \quad \forall A \in L(V).$$

Then there exists $U \in U(\mathbb{C}\Sigma)$ such that

$$B_a = \sum_{b \in \Sigma} U(a,b) A_b.$$

Proof: We have $J(\Phi) = J(\Psi)$, that is,

$$\sum_a \operatorname{vec} A_a (\operatorname{vec} A_a)^\dagger = \sum_a \operatorname{vec} B_a (\operatorname{vec} B_a)^\dagger.$$

Define $u, v \in V \otimes W \otimes \mathbb{C}\Sigma$ by

$$|u\rangle = \sum_a \operatorname{vec} A_a \otimes |a\rangle, \quad |v\rangle = \sum_a \operatorname{vec} B_a \otimes |a\rangle.$$

Then

$$\begin{aligned} \operatorname{Tr}_{\mathbb{C}\Sigma}(|u\rangle\langle u|) &= J(\Phi) \\ &= J(\Psi) \\ &= \operatorname{Tr}_{\mathbb{C}\Sigma}(|v\rangle\langle v|). \end{aligned}$$

By the unitary equivalence of purifications:

$$|v\rangle = \mathbb{1}_{W\otimes V} \otimes U |u\rangle$$

for some $U \in \mathrm{U}(\mathbb{C}\Sigma)$.

Therefore

$$\begin{aligned} \operatorname{vec} B_a &= (\mathbb{1}_{W\otimes V} \otimes \langle a|) \underbrace{|v\rangle}_{\mathbb{1}_{W\otimes V} \otimes U |u\rangle} \\ &= \sum_b (\mathbb{1}_{W\otimes V} \otimes \langle a|)(\operatorname{vec} A_b \otimes U|b\rangle) \\ &= \sum_b U(a,b) \operatorname{vec} A_b. \end{aligned}$$

∎

Unitary equivalence of Stinespring representations

Let $\Phi(C) = \mathrm{Tr}_U(ACA^\dagger)$ and

$\Psi(C) = \mathrm{Tr}_U(BCB^\dagger)$ be such that

$$\Phi(C) = \Psi(C) \quad \forall C \in L(V).$$

Then there exists $U \in \mathcal{U}(U)$ such that

$$B = \mathbb{1}_W \otimes U \, A.$$

Proof: Define

$$A_a = \mathbb{1}_W \otimes \langle a| A \quad \text{and} \quad B_a = \mathbb{1}_W \otimes \langle a| B.$$

Then

$$\underbrace{\mathrm{Tr}_U(ACA^\dagger)}_{\sum_a A_a C A_a^\dagger} = \underbrace{\mathrm{Tr}_U(BCB^\dagger)}_{\sum_a B_a C B_a^\dagger}$$

is equivalent to

$$\sum_a A_a C A_a^\dagger = \sum_a B_a C B_a^\dagger.$$

Hence the result follows from the previous result. $\square$

Characterisations of trace preserving maps

For $\Phi \in T(V, W)$ the following are equivalent.

1) Φ is a trace preserving map.

2) Φ^{+} is a unital map.

3) $\mathrm{Tr}_W J(\Phi) = \mathbb{1}_V$.

4) There exists $\{A_a, B_a \in L(V, W)\}_{a \in \Sigma}$ such that

$$\Phi(A) = \sum_a A_a A B_a^{+}$$

and

$$\sum_a A_a^{+} B_a = \mathbb{1}_V.$$

5) There exists $A, B \in L(V, W \otimes U)$ such that

$$\Phi(C) = \mathrm{Tr}_U (A C B^{+})$$

and $A^{+} B = \mathbb{1}_V$.

Proof: $(1 \Rightarrow 2)$: We have

$$\begin{aligned} \langle \mathbb{1}_V, A \rangle &= \mathrm{Tr}(A) \\ &= \mathrm{Tr}(\Phi(A)) \\ &= \langle \mathbb{1}_W, \Phi(A) \rangle \\ &= \langle \Phi^{+}(\mathbb{1}_W), A \rangle . \end{aligned}$$

Therefore $\Phi^{+}(\mathbb{1}_W) = \mathbb{1}_V$.

$(2 \Rightarrow 1)$: Similar.

$(2 \Rightarrow 4)$: Kraus representation for Φ exists by the Corollary above.

We have $\Phi^+(A) = \sum_a A_a^+ A B_a$.

In particular, $\mathbb{1}_V = \Phi^+(\mathbb{1}_W) = \sum_a A_a^+ B_a$.

$(4 \Rightarrow 2)$: Similar.

$(2 \Leftrightarrow 5)$: Follows from

$$\Phi^+(C) = A^+ (C \otimes \mathbb{1}_U) B.$$

$(1 \Rightarrow 3)$: For $V = \mathbb{C}\Gamma$ we have

$$\mathrm{Tr}_W(J(\Phi)) = \sum_{a,b\in\Gamma} \underbrace{\mathrm{Tr}(\Phi(|a\rangle\langle b|))}_{\delta_{a,b} \text{ since } \Phi \text{ preserves trace}} |a\rangle\langle b|$$

$$= \sum_a |a\rangle\langle a| = \mathbb{1}_V.$$

$(3 \Rightarrow 1)$: We have

$$\sum_a |a\rangle\langle a| = \mathbb{1}_V = \mathrm{Tr}_W(J(\Phi))$$

$$= \sum_{a,b} \mathrm{Tr}(\Phi(|a\rangle\langle b|))\, |a\rangle\langle b|$$

which implies that

$$\mathrm{Tr}(\Phi(|a\rangle\langle b|)) = \delta_{a,b} = \mathrm{Tr}(|a\rangle\langle b|)$$ ☐

Cor: The following are equivalent.

1) Φ is a channel

2) $J(\Phi) \in \mathrm{Pos}(W \otimes V)$ and $\mathrm{Tr}_W(J(\Phi)) = \mathbb{1}_V$.

3) There exists $\{A_a \in L(V,W)\}_{a\in\Sigma}$ such that

$$\Phi(A) = \sum_a A_a A A_a^+$$

and

$$\sum_a A_a^+ A_a = \mathbb{1}_V.$$

4) There exists $A \in L(V, W \otimes U)$ such that

$$\Phi(C) = \mathrm{Tr}_U (A C A^\dagger)$$

and $A^\dagger A = \mathbb{1}_V$, i.e., $A \in U(V, W \otimes U)$.

Another important consequence is that $C(V, W)$ is convex and compact:

We have a linear bijection

$$J: T(V, W) \longrightarrow L(W \otimes V)$$

using which we can write

$$C(V,W) = J^{-1} \{ C \in \mathrm{Pos}(W \otimes V) : \underbrace{\mathrm{Tr}_W C = \mathbb{1}_V}_{\text{convex and closed.}} \}$$

It is also bounded since

$$\| C \|_1 = \mathrm{Tr}\, C = \mathrm{Tr}_V \mathrm{Tr}_W C$$
$$= \mathrm{Tr}_V \mathbb{1}_V$$
$$= \dim V.$$

<u>Aside:</u>

A subset $X \subset \mathbb{R}^n$ is convex if

$$\lambda u + (1-\lambda) v \in X \quad \forall u, v \in X, \; \lambda \in [0,1].$$

Pro: Let $u \in V \otimes W$ and $P \in \mathrm{Pos}(V \otimes U)$ be such that

$$\mathrm{Tr}_W |u\rangle\langle u| = \mathrm{Tr}_U P.$$

Then there exists $\Phi \in C(W, U)$ such that

$$\mathbb{1}_{L(V)} \otimes \Phi\, |u\rangle\langle u| = P.$$

Proof: Let U' be such that

$\dim U' \geq \mathrm{rank}\, P$ and $\dim(U \otimes U') \geq \dim W$.

Let $A \in U(W, U \otimes U')$ and $v \in V \otimes U \otimes U'$ be a purification of P.

We have

$$\begin{aligned}
&\mathrm{Tr}_{U \otimes U'}(\mathbb{1}_V \otimes A\, |u\rangle\langle u|\, \mathbb{1}_V \otimes A^{+}) \\
&= \mathrm{Tr}_W |u\rangle\langle u| \\
&= \mathrm{Tr}_U P \\
&= \mathrm{Tr}_{U \otimes U'} |v\rangle\langle v|.
\end{aligned}$$

By unitary equivalence of purifications there exists $B \in U(U \otimes U')$ such that

$$(\mathbb{1}_V \otimes B)(\mathbb{1}_V \otimes A\, |u\rangle) = |v\rangle.$$

Then define $\Phi \in T(W, U)$ as follows:

$$\Phi(C) = \mathrm{Tr}_{U'}((BA)\, C\, (BA)^{+}).$$

This is a channel since $(BA)^{+} BA = \mathbb{1}_W$.

We have

$$\mathbb{1}_{L(V)} \otimes \Phi\left(|u\rangle\langle u|\right)$$

$$= \mathrm{Tr}_{U'}\left(\left(\mathbb{1}_V \otimes BA\right)|u\rangle\langle u|\left(\mathbb{1}_V \otimes BA\right)^{+}\right)$$

$$= \mathrm{Tr}_{U'}\left(|v\rangle\langle v|\right) = P \qquad \square$$

<u>Note:</u> Let $A \in U(W, W')$.

We have

$$\mathrm{Tr}_{W'}\left(\mathbb{1}_V \otimes A \; |a\rangle\langle b| \otimes |c\rangle\langle d| \; \mathbb{1}_V \otimes A^{+}\right)$$

$$= |a\rangle\langle b| \otimes \underbrace{\underbrace{\mathrm{Tr}_{W'}\left(A\,|c\rangle\langle d|\,A^{+}\right)}_{\langle d|A^{+}A|c\rangle}}_{\langle d|c\rangle}$$

$$= \mathrm{Tr}_{W}\left(|a\rangle\langle d| \otimes |c\rangle\langle d|\right).$$

<u>Ex</u> The channel $\Delta \in C(\mathbb{C}^\Sigma)$

$$\Delta(A) = \sum_{a\in\Sigma} A(a,a)\, |a\rangle\langle a|$$

is called the completely dephasing channel.

1) Natural representation:

$$K(\Delta)\,|ab\rangle = \text{vec}\left(\Delta(|a\rangle\langle b|)\right)$$

$$= \begin{cases} |a\rangle\langle a| & a=b \\ 0 & \text{o/w}, \end{cases}$$

that is, $K(\Delta) = \sum_a \underbrace{|aa\rangle\langle aa|}_{\underbrace{|a\rangle\langle a|}_{A_a} \otimes \underbrace{|a\rangle\langle a|}_{\overline{B_a}}}$.

2) Choi representation:

$$J(\Delta) = \sum_{a,b} \Delta(|a\rangle\langle b|) \otimes |a\rangle\langle b|$$

$$= \sum_a |a\rangle\langle a| \otimes |a\rangle\langle a|$$

$$= \sum_a \underbrace{|aa\rangle}_{\text{vec } A_a} \underbrace{\langle aa|}_{(\text{vec } B_a)^\dagger}.$$

We have

$$\langle v, J(\Delta) v\rangle = \sum_a |\langle v|aa\rangle|^2 \geq 0,$$

hence Δ is a channel. (Trace preserving since diagonal is preserved)

3) Kraus representation:

$$\Delta(A) = \sum_a \underbrace{|a\rangle\langle a|}_{A_a} A \underbrace{|a\rangle\langle a|}_{B_a^\dagger}$$

4) Stinespring representation

$$\Delta(C) = \text{Tr}_{\mathbb{C}^\Sigma}(A C A^\dagger)$$

where $A = \sum_a \underbrace{|a\rangle\langle a|}_{A_a} \otimes |a\rangle.$

Measurements

A measurement is a function

$$\mu : \Sigma \to \mathrm{Pos}(\mathcal{V})$$

set of measurement outcomes

such that

$$\sum_{a \in \Sigma} \mu(a) = \mathbb{1}_{\mathcal{V}}.$$

measurement operators.

A channel $\Phi \in C(\mathcal{V}, \mathcal{W})$ is called a quantum-to-classical channel if

$$\Phi = \Delta \Phi.$$

Pro: The following are equivalent.

1) For every quantum-to-classical $\Phi \in C(V, \mathbb{C}\Gamma)$ there exists a unique measurement $\mu : \Gamma \longrightarrow \mathrm{Pos}(V)$ such that

$$\Phi(A) = \sum_a \langle \mu(a), A \rangle |a\rangle\langle a|.$$

2) For every measurement $\mu : \Gamma \longrightarrow \mathrm{Pos}(V)$ the linear map Φ above is a quantum-to-classical channel.

Proof: $(1 \Rightarrow 2)$: We have

$$\begin{aligned}\Phi(A) &= \Delta \Phi(A) \\ &= \sum_a \langle |a\rangle\langle a|, \Phi(A) \rangle |a\rangle\langle a| \\ &= \sum_a \langle \Phi^\dagger(|a\rangle\langle a|), A \rangle |a\rangle\langle a|.\end{aligned}$$

Then define

$$\mu : \Gamma \longrightarrow \mathrm{Pos}(V)$$

by $\mu(a) = \Phi^\dagger(|a\rangle\langle a|)$.

$\Phi^\dagger$ is positive and unital since Φ is positive and preserves trace.

Therefore

$$\begin{aligned}\sum_a \Phi^\dagger(|a\rangle\langle a|) &= \Phi^\dagger\left(\sum_a |a\rangle\langle a|\right) \\ &= \Phi^\dagger(\mathbb{1}_V) \\ &= \mathbb{1}_V.\end{aligned}$$

If ν is another measurement satisfying $\Phi(A) = \sum_a \langle \nu(a), A \rangle |a\rangle\langle a|$ then

$$\sum_a \langle \mu(a) - \nu(a), A \rangle |a\rangle\langle a| = 0$$

for all A, which implies

$$\langle \mu(a) - \nu(a), A \rangle = 0, \ \forall A \implies \mu(a) = \nu(a).$$

(2 ⟹ 1): The Choi representation is given by

$$J(\Phi) = \sum_{a,b} \Phi(|a\rangle\langle b|) \otimes |a\rangle\langle b|$$

$$= \sum_{a,b} \sum_c \langle \mu(c), |a\rangle\langle b| \rangle |c\rangle\langle c| \otimes |a\rangle\langle b|$$

$$= \sum_c |c\rangle\langle c| \otimes \sum_{a,b} \underbrace{\langle \mu(c), |a\rangle\langle b| \rangle}_{\langle |a\rangle\langle b|, \overline{\mu(c)} \rangle} |a\rangle\langle b|$$

$$= \sum_c \underbrace{|c\rangle\langle c| \otimes \overline{\mu(c)}}_{\text{positive} \implies \text{sum is positive}}$$

Also

$$\mathrm{Tr}_W J(\Phi) = \sum_c \mathrm{Tr}(|c\rangle\langle c|) \overline{\mu(c)}$$

$$= \sum_c \overline{\mu(c)}$$

$$= \overline{\mathbb{1}_V} = \mathbb{1}_V.$$

Φ is quantum-to-classical since $\Phi(A)$ is diagonal for all $A \in L(V)$. $\boxtimes$

As a consequence the set of measurements

$$\mu : \Gamma \longrightarrow \mathrm{Pos}(V)$$

can be identified with the set of quantum-to-classical channels:

$$\{ \Delta \Phi : \Phi \in C(V, \mathbb{C}\Sigma) \} \subset C(V, \mathbb{C}\Sigma).$$

This is precisely the image of

$$\Delta : C(V, \mathbb{C}\Sigma) \longrightarrow C(V, \mathbb{C}\Sigma)$$

sending Φ to $\Delta\Phi$.

Hence compact and convex.

<u>Partial measurements</u>

Let $\mu : \Gamma \longrightarrow \mathrm{Pos}(V)$ be a measurement. The partial measurement associated to μ is the channel

$$\Phi : L(V \otimes W) \longrightarrow L(\mathbb{C}\Lambda \otimes W)$$

defined by

$$\Phi(A) = \sum_a |a\rangle\langle a| \otimes \mathrm{Tr}_V \left(\mu(a) \otimes \mathbb{1}_W A \right).$$

A measurement $\mu : \Gamma \to \mathrm{Pos}(V)$ is called projective if $\mu(a) \in \mathrm{Proj}(V)$, $\forall\, a \in \Gamma$.

<u>Pro</u> : For a projective measurement μ the set $\{\mu(a) : a \in \Gamma\}$ is orthogonal.

<u>Proof</u> : We have

$$\mathbb{1}_V = \Big(\sum_a \mu(a) \Big)^2$$

$$= \underbrace{\sum_a \mu(a)}_{\mathbb{1}_V} + \underbrace{\sum_{a \neq b} \mu(a)\mu(b)}_{0_V}$$

Taking trace of $\sum_{a\neq b} \mu(a)\mu(b) = 0_V$ we get

$$0 = \sum_{a \neq b} \mathrm{Tr}(\mu(a)\mu(b))$$

$$= \sum_{a\neq b} \underbrace{\langle \mu(a), \mu(b) \rangle}_{\in \mathbb{R}_{\geq 0}}$$

Therefore $\langle \mu(a), \mu(b) \rangle = 0 \quad \forall\ a \neq b$. ☐

<u>Ex</u> : For $V = \mathbb{C}^\Sigma$ we have the projective measurement

$$\mu(a) = |a\rangle\langle a|.$$

The associated channel

$$\Phi(\rho) = \sum_a \underbrace{\langle \mu(a), \rho \rangle}_{\langle a|\rho|a\rangle} |a\rangle\langle a|$$

Naimark's theorem

Let $\mu : \Gamma \to \mathrm{Pos}(V)$ be a measurement. There exists $A \in U(V, V \otimes \mathbb{C}\Gamma)$ such that

$$\mu(a) = A^\dagger (\mathbb{1}_V \otimes |a\rangle\langle a|) A$$

Moreover, for a unit vector $u \in \mathbb{C}\Gamma$ there exists a projective measurement

$$\nu : \Gamma \to \mathrm{Pos}(V \otimes \mathbb{C}\Gamma)$$

such that

$$\langle \nu(a), C \otimes |u\rangle\langle u| \rangle = \langle \mu(a), C \rangle.$$

Proof: Define

$$A = \sum_b \sqrt{\mu(b)} \otimes |b\rangle.$$

Then

$$A^\dagger (\mathbb{1}_V \otimes |a\rangle\langle a|) A = \mu(a)$$

and

$$A^\dagger A = \sum_b \mu(b) = \mathbb{1}_V.$$

Let $B \in U(V \otimes \mathbb{C}\Gamma)$ be such that

$$B(\mathbb{1}_V \otimes |u\rangle) = A$$

Define $\nu(a) = B^\dagger (\mathbb{1}_V \otimes |a\rangle\langle a|) B$.

Then

$$\langle \nu(a), C \otimes |u\rangle\langle u| \rangle =$$

$$= \mathrm{Tr}\left(B^\dagger (\mathbb{1}_V \otimes |a\rangle\langle a|) B \, C \otimes \mathbb{1}_{\mathbb{C}\Gamma} \, \mathbb{1}_V \otimes |u\rangle \, \mathbb{1}_V \otimes \langle u| \right)$$

$$= \mathrm{Tr}\Big(\underbrace{\mathbb{1}_V \otimes \langle u | \, B^\dagger}_{A^\dagger} \; \mathbb{1}_V \otimes |a\rangle\langle a| \; \underbrace{B \, \mathbb{1}_V \otimes |u\rangle}_{A} \; C \Big)$$

$$= \mathrm{Tr}\left(A^\dagger \; \mathbb{1}_V \otimes |a\rangle\langle a| \; A \; C \right)$$

$$= \langle \mu(a), C \rangle \qquad \blacksquare$$

Pauli channels (Weyl covariant channels)

Let $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$ denote the additive group of integers, modulo d.

We define two operators in $U(\mathbb{C}\mathbb{Z}_d)$

$$X = \sum_a |a+1\rangle\langle a|$$

$$Z = \sum_a \omega^a |a\rangle\langle a|$$

where $\omega = e^{2\pi i/d}$.

The Pauli operator (Weyl operator) associated to a pair $(a,b) \in \mathbb{Z}_d^2$ is defined by

$$T_{ab} = \sqrt{\omega}^{\,ab} \underbrace{X^a Z^b}_{\sum_c \omega^{bc} |a+c\rangle\langle c|}.$$

Ex: For $d=2$ we have

$$T_{00} = \mathbb{1} \qquad T_{10} = X$$

$$T_{01} = Z \qquad Y_{11} = Y = iXZ.$$

Lem:

1) $T_{ab} T_{ef} = \omega^{be-af} T_{ef} T_{ab}$
2) $T_{ab}^d = \mathbb{1}$
3) $\mathrm{Tr}(T_{ab}) = d\, \delta_{ab,00}$
4) $T_{ab} T_{ef} = \sqrt{\omega}^{\,be-af} T_{a+e,b+f}$
5) $T_{ab}^{-1} = T_{-a,-b} = T_{ab}^\dagger$.

Proof: (1) We have

$$T_{ab} T_{ef} = \sqrt{\omega}^{ab} X^a Z^b \sqrt{\omega}^{ef} X^e Z^f$$

$$ZX|a\rangle = Z|a+1\rangle = \omega^{a+1}|a+1\rangle$$

$$XZ|a\rangle = \omega^a X|a\rangle = \omega^a |a+1\rangle$$

$$\Rightarrow \quad ZX = \omega XZ.$$

Then $Z^b X^e = \omega^{be} X^e Z^b$

$$= \sqrt{\omega}^{ab+ef} \omega^{be} \underbrace{X^{a+e} Z^{b+f}}_{X^e \underbrace{X^a Z^f}_{\omega^{-ad} Z^f X^a} Z^b}$$

$$= \omega^{be-af} \sqrt{\omega}^{ef} X^e Z^f \sqrt{\omega}^{ab} X^a Z^b$$

$$= \omega^{be-af} T_{ef} T_{ab}$$

(2) We have

$$T_{ab}^d = \left(\sqrt{\omega}^{ab} X^a Z^b\right)^d$$

$$= \underbrace{(X^a Z^b)(X^a Z^b) \cdots (X^a Z^b)}_{d}$$

$$= \underbrace{\underbrace{\omega^{b(d-1)a} \omega^{b(d-2)a} \cdots \omega^{ba}}_{\omega^{ba(d-1)d/2}}}_{1} \underbrace{X^{da}}_{\mathbb{1}} \underbrace{Z^{db}}_{\mathbb{1}} = \mathbb{1}.$$

(3) We have

$$\mathrm{Tr}(T_{ab}) = \mathrm{Tr}\left(\sqrt{\omega}^{ab} X^a Z^b\right)$$

$$= \sqrt{\omega}^{ab}\, \mathrm{Tr}\left(\Big(\sum_e |e+1\rangle\langle e|\Big)^a \Big(\sum_f \omega^f |f\rangle\langle f|\Big)^b\right)$$

$$= \sqrt{\omega}^{ab}\, \mathrm{Tr}\left(\sum_e |e+a\rangle\langle e| \sum_f \omega^{fb} |f\rangle\langle f|\right)$$

$$= \sqrt{\omega}^{ab} \sum_e \omega^{eb}\, \underbrace{\mathrm{Tr}\left(|e+a\rangle\langle e|\right)}_{\delta_{a,0}}$$

$$= \begin{cases} d & (a,b) = (0,0) \text{ in } \mathbb{Z}_d^2, \\ 0 & \text{otherwise.} \end{cases}$$

Note $a=0$, $b \neq 0$:

$$\sum_e \omega^{eb} = \sum_e (\omega^b)^e = \sum_e \Big(\underbrace{e^{2\pi i b/d}}_{\mu}\Big)^e = 0$$

If $\mu \in U(\mathbb{C})$ such that $\mu \neq 1$ & $\mu^d = 1$ then

$$\sum_{e=1}^{d} \mu^e = 0$$

since $\mu^d - 1 = 0$ implies $(\mu - 1)\sum_e \mu^e = 0$

(4) We have

$$T_{ab} T_{ef} = \sqrt{\omega}^{ab+ef} \omega^{be} X^{a+e} Z^{b+f}$$

On the other hand,

$$T_{a+e,b+f} = \underbrace{\sqrt{\omega}^{(a+e)(b+f)}}_{\sqrt{\omega}^{ab+af+eb+ef}} X^{a+e} Z^{b+f}$$

Therefore

$$T_{ab} T_{ef} = \omega^{be} \sqrt{\omega}^{-af-eb} T_{a+e,b+f}$$

$$= \sqrt{\omega}^{be-af} T_{a+e,b+f}.$$

(5) We have

$$T_{ab} T_{-a,-b} = \sqrt{\omega}^{-ba+ab} T_{0,0}$$

$$= \mathbb{1}$$

Note that $X^{+} = X^{-1}$ and $Z^{+} = Z^{-1}$. (exercise)

Therefore

$$T_{ab}^{+} = \sqrt{\omega}^{-ab} \underbrace{Z^{-b} X^{-a}}_{\omega^{ba} X^{-a} Z^{-b}}$$

$$= \sqrt{\omega}^{ab} X^{-a} Z^{-b}$$

$$= T_{-a,-b}.$$

$\square$

Cor: $\left\{ \frac{1}{\sqrt{d}} T_{ab} : (a,b) \in \mathbb{Z}_d^2 \right\}$ is an orthonormal basis for $L(\mathbb{C}\mathbb{Z}_d)$.

Proof: By the Lemma:

$$\langle T_{ab}, T_{ef} \rangle = \mathrm{Tr}\left(T_{ab}^{\dagger} T_{ef} \right)$$
$$= \mathrm{Tr}\left(T_{-a,-b} T_{ef} \right)$$
$$= \mathrm{Tr}\left(\sqrt{\omega}^{-be+af} T_{-a+e,-b+f} \right)$$
$$= d\, \delta_{ab,ef}$$

$\boxtimes$

A channel $\Phi \in C(\mathbb{C}\mathbb{Z}_d)$ is called a Pauli channel (Weyl covariant) if

$$\Phi(T_{ab} A T_{ab}^{\dagger}) = T_{ab} \Phi(A) T_{ab}^{\dagger}$$

for all $(a,b) \in \mathbb{Z}_d^2$.

Pro: The following are equivalent.

1) Φ is a Pauli channel.

2) $\Phi(T_{ab}) = A(a,b)\, T_{ab}$, $A(a,b) \in \mathbb{C}$.

3) $\Phi(A) = \sum_{a,b} B(a,b)\, T_{ab} A T_{ab}^{+}$,

where $B(a,b) \in \mathbb{R}_{\geq 0}$ and $\sum_{a,b} B(a,b) = 1$.

Proof: $(1 \Rightarrow 2)$: We have

$$T_{ab}^{+}\, \Phi(T_{ab})\, T_{ef} = T_{ab}^{+}\, T_{ef}^{+}\, T_{ef}\, \Phi(T_{ab})\, T_{ef}$$

$$= T_{ab}^{+}\, T_{ef}^{+}\, T_{ef}\, \Phi(T_{ab})\, T_{ef}^{+}$$

$$= \underbrace{T_{ab}^{+}\, T_{ef}^{+}}_{\omega^{-(fa-eb)}\, T_{ef}^{+} T_{ab}^{+}}\, \Phi(\underbrace{T_{ef}\, T_{ab}\, T_{ef}^{+}}_{\omega^{fa-eb}\, T_{ab}})$$

$$= T_{ef}^{+}\, T_{ab}^{+}\, \Phi(T_{ab}),$$

i.e., $[T_{ab}^{+}\, \Phi(T_{ab}), T_{ef}^{+}] = 0$.

Notation: $[A, B] = AB - BA$.

Note that $[A, B] = 0 \iff \underbrace{AB = BA}_{\text{operators commute}}$.

Since $\{T_{ef}\}_{e,f}$ is a basis this implies that

$$T_{ab}^{+}\, \Phi(T_{ab}) = A(a,b)\, \mathbb{1}$$

for some $A(a,b) \in \mathbb{C}$.

Thus

$$\Phi(T_{ab}) = A(a,b)\, T_{ab}.$$

(2 ⇒ 1) : We have

$$\Phi(\underbrace{T_{ab} T_{ef}}_{\omega^{be-af} T_{ef} T_{ab}} T_{ab}^{\dagger}) = \omega^{be-af} \underbrace{\Phi(T_{ef})}_{A(e,f)\, T_{ef}}$$

$$= A(e,f)\, T_{ab} T_{ef} T_{ab}^{\dagger}$$

$$= T_{ab}\, \Phi(T_{ef})\, T_{ab}^{\dagger} .$$

For $A \in L(V)$ we have

$$\Phi(T_{ab} \underbrace{A}_{\sum_{e,f} \alpha_{ef} T_{ef}} T_{ab}^{\dagger}) = \sum_{e,f} \alpha_{ef} \underbrace{\Phi(T_{ab} T_{ef} T_{ab}^{\dagger})}_{T_{ab}\, \Phi(T_{ef})\, T_{ab}^{\dagger}}$$

$$= T_{ab}\, \Phi\Big(\sum_{e,f} \alpha_{ef} T_{ef}\Big)\, T_{ab}^{\dagger}$$

$$= T_{ab}\, \Phi(A)\, T_{ab}^{\dagger} .$$

(3 ⇒ 2) : We have

$$\Phi(T_{ef}) = \sum_{a,b} B(a,b) \underbrace{T_{ab} T_{ef} T_{ab}^{\dagger}}_{\omega^{be-af} T_{ef}}$$

$$= \sum_{a,b} B(a,b)\, \omega^{be-af}\, T_{ef}$$

$$= A(e,f)\, T_{ef}$$

where

$$A(e,f) = \sum_{a,b} B(a,b)\, \omega^{be-af}\, T_{ef} .$$

Define the operator:

$$F = \frac{1}{\sqrt{d}} \sum_{a,b} \omega^{ab} |a\rangle\langle b|.$$

called the Fourier transform.

Then we have

$$d\, F^{\dagger} B F = \sum_{a,b} \omega^{-ab} |b\rangle\langle a| B \sum_{e,f} \omega^{ef} |e\rangle\langle f|$$

$$= \sum_{a,b,e,f} \omega^{ef-ab} B(a,e)\, |b\rangle\langle f|$$

$$= \sum_{b,f} \sum_{a,e} \omega^{ef-ab} B(a,e)\, |b\rangle\langle f|$$

$$= \sum_{b,f} A(f,b)\, |b\rangle\langle f|$$

$$= A^{T}.$$

That is, $d\, F^{\dagger} B F = A^{T}$.

$(2 \Rightarrow 3)$: Similar, follows from

$$B = \frac{1}{d} F A^{T} F^{\dagger}.$$

The Choi representation of Φ is given by

$$J(\Phi) = \sum_{a,b} B(a,b)\, \underbrace{\operatorname{vec} T_{ab}}_{\text{basis for } \mathbb{C}\mathbb{Z}_d \otimes \mathbb{C}\mathbb{Z}_d} (\operatorname{vec} T_{ab})^{\dagger}.$$

$J(\Phi)$ positive implies that $B(a,b) \in \mathbb{R}_{\geq 0}$.

Trace preservation implies that

$$\underbrace{\mathrm{Tr}(\Phi(A))}_{\mathrm{Tr}\, A} = \sum_{a,b} B(a,b)\, \mathrm{Tr}(A)$$

which gives $\sum_{a,b} B(a,b) = 1$.

Ex : 1) Completely depolarizing channel

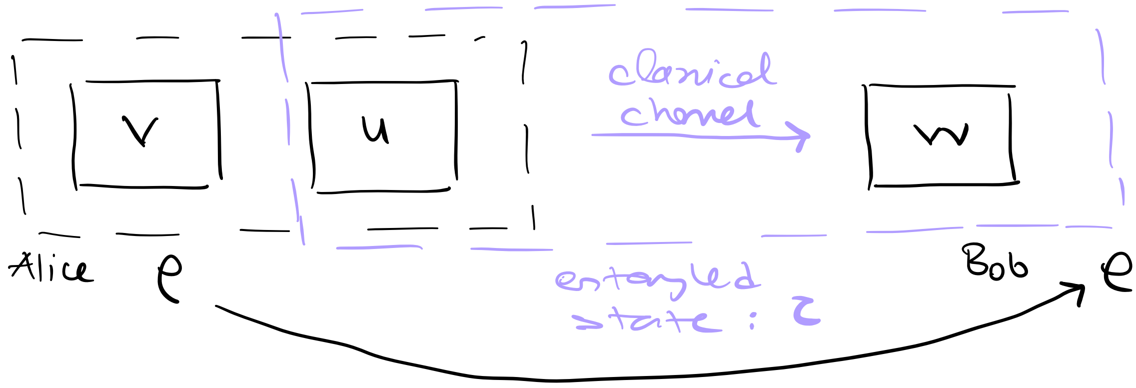
$$\Omega(A) = \frac{1}{d^2} \sum_{a,b} T_{ab}\, A\, T_{ab}^{+} .$$

2) Completely dephasing channel

$$\Delta(A) = \frac{1}{d} \sum_{a} T_{0a}\, A\, T_{0a}^{+} .$$

Teleportation

Teleportation is a basic protocol for transmitting quantum information:



$$\Phi = \mathbb{1}_{L(\mathbb{C}\mathbb{Z}_d)}$$

Here $V = U = W = \mathbb{C}\mathbb{Z}_d$.

We fix

1) a state τ in $\mathrm{Den}(U \otimes W)$

$$\tau = \frac{1}{d} \operatorname{vec} \mathbb{1} \, (\operatorname{vec} \mathbb{1})^{+}$$

$$= \frac{1}{d} \sum_{b,c} |b\rangle\langle c| \otimes |b\rangle\langle c|$$

2) a measurement

$$\mu : \mathbb{Z}_d^2 \longrightarrow \mathrm{Pos}(V \otimes U)$$

where

$$\mu(ab) = \frac{1}{d} \operatorname{vec} T_{ab} \, (\operatorname{vec} T_{ab})^{+}$$

$$= \frac{1}{d} \sum_{e,f} \omega^{be-bf} |a+e\rangle\langle a+f| \otimes |e\rangle\langle f|$$

Recall $T_{ab} = \sqrt{\omega}^{ab} \sum_{e} \omega^{be} |a+e\rangle\langle e|$

3) a channel for each $(a,b) \in \mathbb{Z}_d^2$

$$\Phi_{ab}(\rho) = T_{ab}\, \rho\, T_{ab}^{+}.$$

Define the channels

i) $\Phi_1 : L(V) \longrightarrow L(V \otimes U \otimes W)$

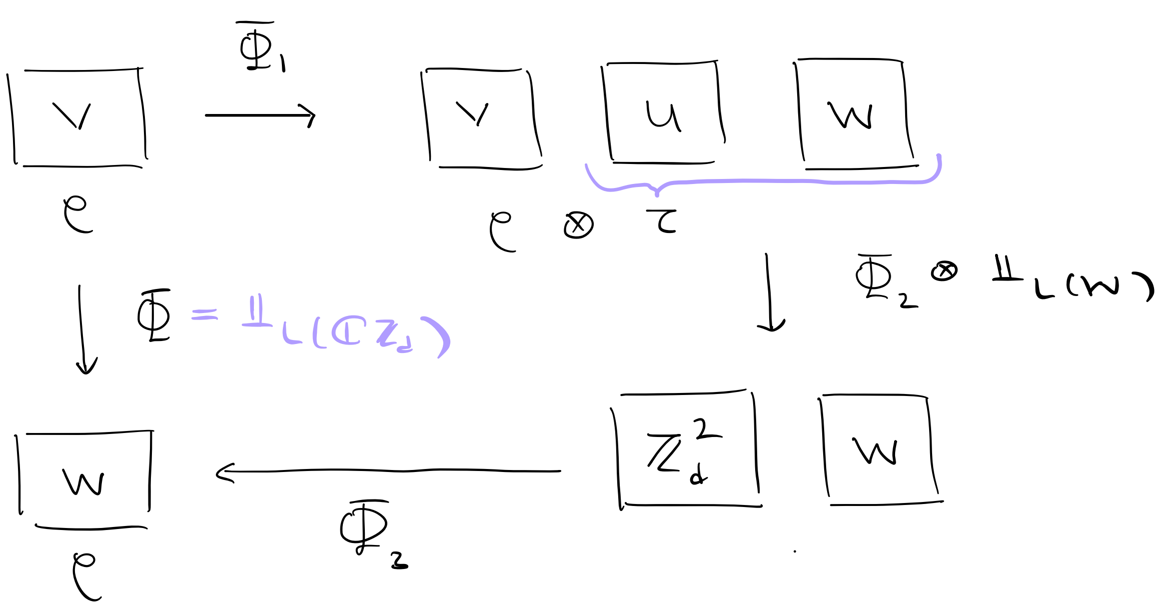
$$\Phi_1(\rho) = \rho \otimes \tau$$

ii) $\Phi_2 : L(V \otimes U) \longrightarrow L(\mathbb{C}\mathbb{Z}_d^2)$

$$\Phi_2(\sigma) = \sum_{a,b} \mathrm{Tr}_{V\otimes U}(\mu(ab)\, \sigma)\, |ab\rangle\langle ab|$$

iii) $\Phi_3 : L(\mathbb{C}\mathbb{Z}_d^2 \otimes W) \longrightarrow L(\mathbb{C}\mathbb{Z}_d^2 \otimes W)$

$$\Phi_3\Big(\sum_{ab} p(ab)\, |ab\rangle\langle ab| \otimes \gamma\Big) = \sum_{ab} p(ab)\, \underbrace{\Phi_{ab}(\gamma)}_{\Phi_3(|ab\rangle\langle ab| \otimes \gamma)}$$



Lem: $\mathrm{vec}\, \mathbb{1} (\mathrm{vec}\, \mathbb{1})^\dagger = \frac{1}{d} \sum_{a,b} \overline{T_{ab}} \otimes T_{ab}$.

Proof: We have

$$\frac{1}{d} \sum_{a,b} \overline{T_{ab}} \otimes T_{ab}$$

$$= \underbrace{\frac{1}{d} \sum_{a,b} \sum_{c,e} \omega^{be-bc}}_{} |a+c\rangle\langle c| \otimes |a+c\rangle\langle c|$$

$$\frac{1}{d} \sum_{b} \omega^{b(e-c)} = \delta_{e,c}$$

$$T_{ab} = \sqrt{\omega}^{ab} \sum_{e} \omega^{be} |a+e\rangle\langle e|$$

$$\overline{T_{ab}} = \sqrt{\omega}^{-ab} \sum_{c} \omega^{-bc} |a+c\rangle\langle c|$$

$$= \sum_{a,c} |\underbrace{a+c}_{f}\rangle\langle c| \otimes |a+c\rangle\langle c|$$

$$= \sum_{f,c} |f\rangle\langle c| \otimes |f\rangle\langle c|.$$

$\square$

Pro: We have

$$\Phi = \Phi_3 \circ \Phi_2 \otimes \mathbb{1}_{L(W)} \circ \Phi_1$$

satisfies $\Phi = \mathbb{1}_{L(V)}$

i.e., $\Phi(A) = A, \quad \forall A \in L(V)$.

Proof: Since $\{T_{ab}\}_{a,b}$ is a basis it suffices to show $\Phi(T_{ab}) = T_{ab}$ $\forall\, a,b$.

We have

$$\Phi(T_{ab}) = \Phi_3 \circ \Phi_2 \otimes \mathbb{1}_{L(W)} \circ \Phi_1 (T_{ab})$$

$\Phi_1(T_{ab}) = T_{ab} \otimes \tau$

$$\sum_{c,e} |ce\rangle\langle ce| \otimes \mathrm{Tr}_{V \otimes U} (\mu|ce)\, T_{ab} \otimes \tau)$$

$\mu|ce) = \frac{1}{d} \operatorname{vec} T_{ce} (\operatorname{vec} T_{ce})^{\dagger}$, $\quad \tau = \frac{1}{d^2} \sum_{f,g} \overline{T}_{f,g} \otimes T_{f,g}$

$$= \frac{1}{d^3} \sum_{\substack{c,e\\ f,g}} \langle \operatorname{vec} T_{ce} (\operatorname{vec} T_{ce})^{\dagger}, T_{ab} \otimes \overline{T}_{fg} \rangle\, T_{ce} T_{fg} T_{ce}^{\dagger}$$

$\mathrm{Tr}(\operatorname{vec} T_{ce} (\operatorname{vec} T_{ce})^{\dagger}\, T_{ab} \otimes \overline{T}_{fg})$

We use

$$(A_0 \otimes A_1) \operatorname{vec}(B) = \operatorname{vec}(A_0 B A_1^{T})$$

$$= \frac{1}{d^3} \sum_{\substack{c,e\\ f,g}} \mathrm{Tr}((\operatorname{vec} T_{ce})^{\dagger} \operatorname{vec}(T_{ab} T_{ce} T_{fg}^{\dagger}))\, T_{ce} T_{fg} T_{ce}^{\dagger}$$

$$= \frac{1}{d^3} \sum_{\substack{c,e\\ f,g}} \mathrm{Tr}(T_{ce}^{\dagger} T_{ab} T_{ce} T_{fg}^{\dagger})\, T_{ce} T_{fg} T_{ce}^{\dagger}$$

(1) (2) commute (3); phases cancel; commute

$$= \frac{1}{d} \sum_{f,g} \mathrm{Tr}(T_{fg}^{\dagger} T_{ab})\, T_{fg} = T_{ab}.$$

$\langle T_{fg}, T_{ab} \rangle = d\, \delta_{fg,ab}$

That is,

$$\Phi(T_{ab}) = T_{ab} \quad \forall (a,b) \in \mathbb{Z}_d^2 . \qquad \boxed{6}$$

Operational meaning:

$$\Phi_2 \otimes \mathbb{1}_{L(W)} \circ \Phi_1 (T_{ab}) =$$

$$= \frac{1}{d^2} \sum_{\substack{c,e \\ f,g}} |ce\rangle\langle ce| \otimes \omega^{cg-ef} \, d \, \delta_{fg,ab} \, T_{fg}$$

$$= \sum_{c,e} \frac{1}{d^2} |ce\rangle\langle ce| \otimes \omega^{cb-ea} \, T_{ab}$$

For a density operator

$$\rho = \sum_{a,b} \alpha_{ab} T_{ab}$$

we obtain

$$\Phi_2 \otimes \mathbb{1}_{L(W)} \circ \Phi_1 (\rho) =$$

$$\sum_{c,e} \frac{1}{d^2} |ce\rangle\langle ce| \otimes \underbrace{\sum_{a,b} \alpha_{ab} \omega^{cb-ea} T_{ab}}_{\rho_{ce}} .$$

(ensemble of states)

If the outcome of measurement μ is $(c,e) \in \mathbb{Z}_d^2$ then the compound register (with classical states $\mathbb{Z}_d \times \mathbb{Z}_d \times \mathbb{Z}_d$) is in state

$$|ce\rangle\langle ce| \otimes \rho_{ce} .$$

Then Φ_3 makes the final correction:

1) $|ce\rangle\langle ce|$ is discarded.

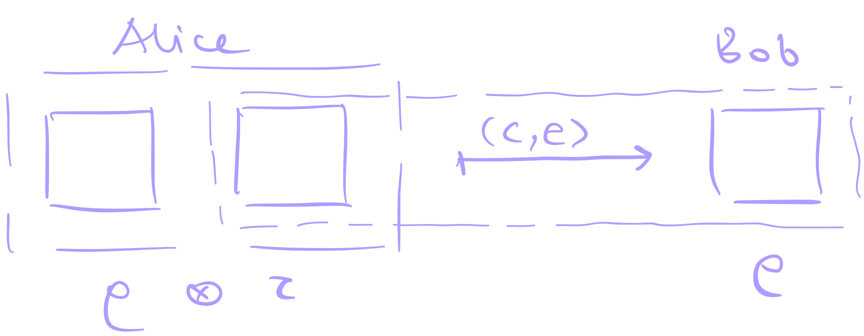
2) ρ_{ce} becomes

$$T_{ce}\, \rho_{ce}\, T_{ce}^{\dagger} = \sum_{a,b} \alpha_{ab}\, \omega^{cb-ea}\, \underbrace{T_{ce} T_{ab}}_{\omega^{ea-cb} T_{ab} T_{ce}} T_{ce}^{\dagger}$$

$$= \sum_{a,b} \alpha_{ab} T_{ab}$$

$$= \rho .$$

after Alice communicates the classical information of the measurement outcome $(c,e) \in \mathbb{Z}_d^2$ to Bob.



Channels on density operators

Choosing $\Sigma \cong \{0, 1, .., |\Sigma| - 1\}$:

we can construct a basis of $L(\mathbb{C}\Sigma)$ consisting of density operators

$$\rho_{ab} = \begin{cases} |a\rangle\langle a| & a = b \\ \frac{1}{2}(|a\rangle + |b\rangle)(\langle a| + \langle b|) & a < b \\ \frac{1}{2}(|a\rangle + i|b\rangle)(\langle a| - i\langle b|) & a > b. \end{cases}$$

Therefore a channel $\Phi \in C(V, W)$ is determined by its restriction to density operators:

$$\Phi : \mathrm{Den}(V) \longrightarrow \mathrm{Den}(W).$$

We have

$$\Phi\left(\sum_i p_i \rho_i\right) = \sum_i p_i \Phi(\rho_i)$$

where $p_i \in \mathbb{R}_{\geq 0}$ and $\sum_i p_i = 1$.

Since $\{\rho_{ab}\}_{a,b}$ is also a basis for $\mathrm{Herm}(V)$ we have

$$\Phi : \mathrm{Her}(V) \longrightarrow \mathrm{Her}(W)$$

$\mathbb{R}$ - linear.

Single qubit channels

A channel $\Phi \in C(\mathbb{C}\mathbb{Z}_2)$ is determined by its restriction to density operators:

$$\Phi : \mathrm{Den}(\mathbb{C}\mathbb{Z}_2) \longrightarrow \mathrm{Den}(\mathbb{C}\mathbb{Z}_2)$$

Pro: $\mathrm{Den}(\mathbb{C}\mathbb{Z}_2) =$

$$\left\{ \rho = \frac{1}{2} \sum_{a,b} r_{ab} T_{ab} \ : \ r_{00} = 1, \ r_{10}^2 + r_{11}^2 + r_{10}^2 \leq 1 \right\}.$$

Proof: Since $\{T_{ab}\}$ is a basis of $L(\mathbb{C}\mathbb{Z}_2)$ we can write

$$\rho = \frac{1}{2} \sum_{a,b} r_{ab} T_{ab}, \quad r_{ab} \in \mathbb{C}.$$

We have

1) $\mathrm{Tr}\, \rho = 1$:

$$1 = \mathrm{Tr}\, \rho = \frac{1}{2} \sum_{a,b} r_{ab} \underbrace{\mathrm{Tr}(T_{ab})}_{2\delta_{ab,00}}$$

$$= r_{00}.$$

2) $\rho \in \mathrm{Pos}(\mathbb{C}\mathbb{Z}_2)$:

Recall

$$T_{00} = \mathbb{1}, \quad T_{10} = X, \quad T_{01} = Z, \quad T_{11} = Y.$$

Then

$$\rho = \frac{1}{2}\begin{pmatrix} r_{00} + r_{01} & r_{10} - i r_{11} \\ r_{10} + i r_{11} & r_{00} - r_{01} \end{pmatrix}$$

and the eigenvalues are given by

$$\lambda_\pm = \frac{1}{2}\left(r_{00} \pm \sqrt{r_{10}^2 + r_{11}^2 + r_{01}^2} \right).$$

Note $\rho \in \mathrm{Pos}(\mathbb{C}^{\mathbb{Z}_2}) \iff \lambda_\pm \in \mathbb{R}_{\geq 0}$.

Combining (1) and (2):

$$\frac{1}{2}\left(1 \pm \sqrt{r_{10}^2 + r_{11}^2 + r_{01}^2} \right) \geq 0$$

$$\iff \quad r_{10}^2 + r_{11}^2 + r_{01}^2 \leq 1. \qquad \square$$

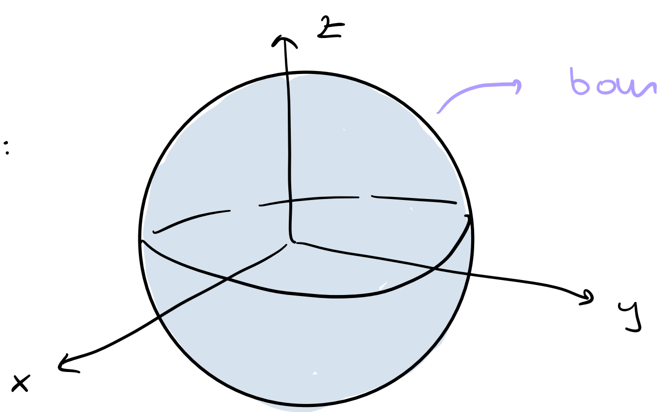
We will identify 00, 10, 11, 01 with 0, 1, 2, 3; respectively:

$$\rho = \frac{1}{2} \sum_{i=0}^{3} r_i \sigma_i$$

$\begin{pmatrix} \sigma_0 = \mathbb{1} \\ \sigma_1 = X \\ \sigma_2 = Y \\ \sigma_3 = Z \end{pmatrix}$

where $r_0 = 1$ and $\sum_{i=1}^{3} r_i^2 \leq 1$.

Picture:



boundary: Bloch sphere

Aside: Eigenvalues of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are the roots of

$$\underbrace{\det(A - \lambda \mathbb{1})} = 0$$

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 + (-a-d)\lambda + ad - bc$$

$$\lambda_\pm = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

We will need the following basic channels:

1) Deterministic bit flip:

$$\Phi_1(\rho) = X \rho X^{+}.$$

2) Deterministic phase flip:

$$\Phi_3(\rho) = Z \rho Z^{+}$$

3) Deterministic bit-phase flip:

$$\Phi_2(\rho) = Y \rho Y^{+}.$$

<u>Thm</u>: Every unital channel $\Phi \in C(\mathbb{C}\mathbb{Z}_2)$, i.e., $\Phi(\mathbb{1}_{\mathbb{C}\mathbb{Z}_2}) = \mathbb{1}_{\mathbb{C}\mathbb{Z}_2}$, is of the form

$$\Phi = \Phi^{U_1} \circ \underbrace{\left(\sum_{i=1}^{3} p_i \Phi_i \right)}_{\text{Pauli channel.}} \circ \Phi^{U_2}$$

where $p_i \in \mathbb{R}_{\geq 0}$ and $\sum_i p_i = 1$

and

$$\Phi^{U}(\rho) = U \rho U^{+}, \quad U \in U(\mathbb{C}\mathbb{Z}_2).$$

<u>Proof</u>: $\Phi : Her(\mathbb{C}\mathbb{Z}_2) \longrightarrow Her(\mathbb{C}\mathbb{Z}_2)$ is $\mathbb{R}$-linear.

Writing

$$A = \frac{1}{2} \sum_{i=0}^{3} r_i \sigma_i, \quad r_i \in \mathbb{R}$$

for a Hermitian operator, Φ can be

expressed as a 4×4 real matrix:

$$\Phi^M = \begin{pmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & & & \\ M_{20} & & M & \\ M_{30} & & & \end{pmatrix}$$

for some real 3×3 matrix M.

We have

1) Φ is trace preserving:

$$\mathrm{Tr}(\Phi(\sigma_i)) = \mathrm{Tr}(\sigma_i)$$

This implies that

$$M_{01} = M_{02} = M_{03} = 0.$$

2) Φ is unital:

This implies that

$$M_{10} = M_{20} = M_{30} = 0.$$

Let $\Phi(A) = \frac{1}{2}\sum_{j=0}^{3} s_j \sigma_j$.

Then (1) $\iff$ $s_0 = 0$ for $A = \frac{1}{2}\sigma_j$ $j = 1,2,3$.

(2) $\iff$ $s_j = 0$ $j = 1,2,3$ for $A = \frac{1}{2}\sigma_0$.

Therefore

$$\Phi^M = \left(\begin{array}{c|c} 1 & 0_{1\times 3} \\ \hline 0_{3\times 1} & M \end{array}\right)$$

By the singular value decomposition:

$$M = O_1 D' O_2$$

where

$$D' = \begin{pmatrix} s_1' & 0 & 0 \\ 0 & s_2' & 0 \\ 0 & 0 & s_3' \end{pmatrix} \qquad s_i' \in \mathbb{R}_{\geq 0}$$

and O_1, O_2 are $\underbrace{\text{orthogonal}}_{O^T = O^{-1}}$ 3×3 matrices.

Any orthogonal matrix O can be written as

$$O = \overbrace{(-1)^a}^{\det O} R \quad , \quad a \in \mathbb{Z}_2$$

where R is a rotation matrix.

Then

$$
\begin{aligned}
M &= O_1 D' O_2 \\
&= (-1)^{a_1} R_1 D' (-1)^{a_2} O_2 \\
&= R_1 (-1)^{a_1+a_2} D' R_2 \\
&= R_1 D R_2
\end{aligned}
$$

where

$$D = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}$$

and $s_i \in \mathbb{R}$.

Note that $\Phi^M = \Phi^{R_1} \circ \Phi^D \circ \Phi^{R_2}$.

Below we will see that there exists $U \in U(\mathbb{C}\mathcal{H}_i)$ such that

$$\Phi^U = \Phi^R.$$

For the rest we will assume

$$M = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}$$

where $s_i \in \mathbb{R}$.

Next we compute the Choi operator of $\Phi = \Phi^M$.

We will use

$$|0\rangle\langle 0| = \frac{1}{2}(\mathbb{1} + Z)$$
$$|0\rangle\langle 1| = \frac{1}{2}(X + iY)$$
$$|1\rangle\langle 0| = \frac{1}{2}(X - iY)$$
$$|1\rangle\langle 1| = \frac{1}{2}(\mathbb{1} - Z).$$

$$|a\rangle\langle b| = \sum_{a,b} \langle T_{ab}, |a\rangle\langle b|\rangle \, T_{ab}.$$

Then we have

$$\Phi(|0\rangle\langle 0|) = \frac{1}{2}(\mathbb{1} + s_3 Z) = \frac{1}{2}\begin{pmatrix} 1+s_3 & 0 \\ 0 & 1-s_3 \end{pmatrix}$$
$$\Phi(|0\rangle\langle 1|) = \frac{1}{2}(s_1 X + i s_2 Y) = \frac{1}{2}\begin{pmatrix} 0 & s_1+s_2 \\ s_1-s_2 & 0 \end{pmatrix}$$
$$\Phi(|1\rangle\langle 0|) = \frac{1}{2}(s_1 X - i s_2 Y) = \frac{1}{2}\begin{pmatrix} 0 & s_1-s_2 \\ s_1+s_2 & 0 \end{pmatrix}$$
$$\Phi(|1\rangle\langle 1|) = \frac{1}{2}(\mathbb{1} - s_3 Z) = \frac{1}{2}\begin{pmatrix} 1-s_3 & 0 \\ 0 & 1+s_3 \end{pmatrix}.$$

The Choi operator is given by

$$J(\Phi) = \frac{1}{2}\sum_{a,b\in\mathbb{Z}_2} \underbrace{\Phi(|a\rangle\langle b|) \otimes |a\rangle\langle b|}_{\text{Kronecker product}}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{nm}B \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1+s_3 & 0 & 0 & s_1+s_2 \\ 0 & 1-s_3 & s_1-s_2 & 0 \\ 0 & s_1-s_2 & 1-s_3 & 0 \\ s_1+s_2 & 0 & 0 & 1+s_3 \end{pmatrix}$$

The eigenvalues of $J(\Phi)$ are given by the eigenvalues of the two blocks:

$$\frac{1}{4}\begin{pmatrix} 1-s_3 & s_1-s_2 \\ s_1-s_2 & 1-s_3 \end{pmatrix} \quad \text{and} \quad \frac{1}{4}\begin{pmatrix} 1+s_3 & s_1+s_2 \\ s_1+s_2 & 1+s_3 \end{pmatrix}$$

$$\lambda_0 = (1+s_1+s_2+s_3)/4$$
$$\lambda_1 = (1+s_1-s_2-s_3)/4$$
$$\lambda_2 = (1-s_1+s_2-s_3)/4$$
$$\lambda_3 = (1-s_1-s_2+s_3)/4.$$

The map Φ is completely positive if and only if $\lambda_i \geq 0$, $\forall i$:

$$1+s_1+s_2+s_3 \geq 0 \qquad 1-s_1+s_2-s_3 \geq 0$$
$$1+s_1-s_2-s_3 \geq 0 \qquad 1-s_1-s_2+s_3 \geq 0,$$

or more compactly

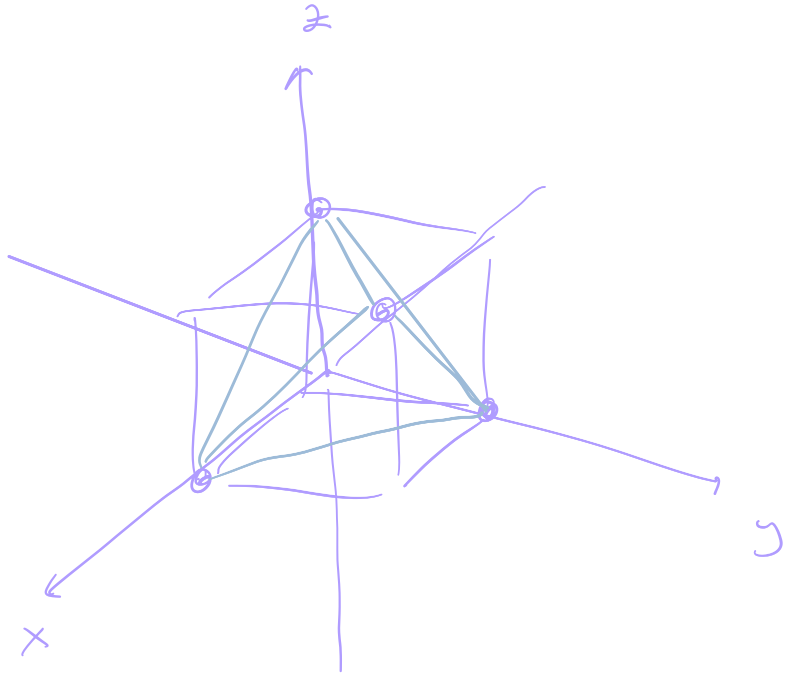
$$1+s_3 \geq |s_1+s_2|$$
$$1-s_3 \geq |s_1-s_2|.$$

(Fujiwara – Algoet conditions)

These inequalities specify a polytope (tetrahedron) in $\mathbb{R}^3$ with vertices

$(1,1,1)$ (identity) $\quad (1,-1,-1)$ (bit flip) $\quad (-1,1,-1)$ (bit-phase flip) $\quad (-1,-1,1)$ (phase flip).

☑



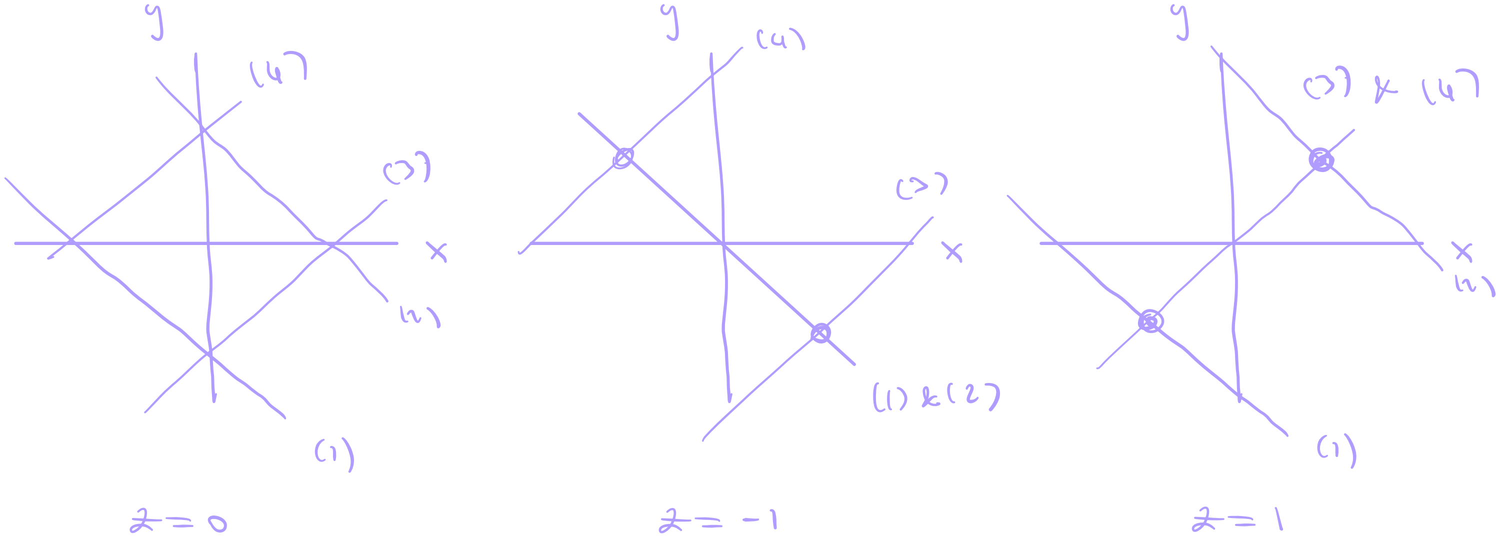
Hyperplanes:

$$z = -1 - x - y \quad (1)$$

$$z = 1 - x - y \quad (2)$$

$$z = 1 + x - y \quad (3)$$

$$z = -1 + x + y \quad (4)$$



Lem: Let R be a 3×3 rotation matrix. Then there exists $U \in U(\mathbb{C}\mathbb{Z}_2)$ such that

$$\Phi^R(\rho) = U\rho U^\dagger .$$

Proof: $U(\mathbb{C}\mathbb{Z}_2)$ acts on $\mathrm{Den}(\mathbb{C}\mathbb{Z}_2)$:

Given $\rho \in \mathrm{Den}(\mathbb{C}\mathbb{Z}_2)$ the operator $U\rho U^\dagger$ is also a density operator:

1) $\mathrm{Tr}(U\rho U^\dagger) = \mathrm{Tr}\,\rho = 1$

2) Positivity:

$\langle v, U\rho U^\dagger v\rangle = \langle U^\dagger v, \rho U^\dagger v\rangle \geq 0.$

We can write

$$U = \sum_a \lambda_a |v_a\rangle\langle v_a|$$

$\lambda_a = e^{i\vartheta_a}$, $\vartheta_a \in \mathbb{R}_{\geq 0}$

$$= e^{iA}$$

where $A \in \mathrm{Her}(\mathbb{C}\mathbb{Z}_2)$:

$$A = \sum_a \vartheta_a |v_a\rangle\langle v_a| .$$

Writing $A = \frac{1}{2}\sum_{i=0}^{3} \alpha_i \sigma_i$

We have

$$U = e^{-i(\alpha_0 \mathbb{1} + \alpha_1 X + \alpha_2 Y + \alpha_3 Z)/2}$$

$$= e^{-i\alpha_0/2} \underbrace{e^{-i(\alpha_1 X + \alpha_2 Y + \alpha_3 Z)/2}}_{V}.$$

We have

$$U \rho U^{\dagger} = V \rho V^{\dagger}.$$

Let us write

$$\rho = \frac{1}{2}\left(\mathbb{1} + \underbrace{r \cdot \sigma}_{\sum_{i=1}^{3} r_i \sigma_i}\right)$$

Then

$$e^{-\frac{i}{2}\alpha\cdot\sigma} \rho \, e^{\frac{i}{2}\alpha\cdot\sigma} = \frac{1}{2}\left(\mathbb{1} + \left(\overbrace{R_{\hat{\alpha}}(|\alpha|)}^{3\times 3 \text{ matrix}}\, r\right)\cdot\sigma\right).$$

where $R_{\hat{\alpha}}(|\alpha|)$ is the rotation matrix rotating about $\underbrace{\hat{\alpha}}$ by angle $|\alpha|$

unit vector in the direction of α.

☑

Exercise

Reference: Geometry of quantum states by Ingemar Bengtsson and Karol Życzkowski

Cor: Every unital channel $\Phi \in C(\mathbb{C}\mathbb{Z}_2)$ is a mixed unitary channel:

$$\Phi(A) = \sum_{a \in \Sigma} p(a)\, U_a A U_a^{\dagger}$$

where $U_a \in U(\mathbb{C}\mathbb{Z}_2)$, $p(a) \in \mathbb{R}_{\geq 0}$ such that $\sum_a p(a) = 1$.

Proof: We have

$$\Phi(A) = \Phi^{U_1} \circ \Phi^{D} \circ \Phi^{U_2}(A)$$

$$= \sum_{i=0} p_i \underbrace{U_1 \sigma_i U_2}_{V_i} A \underbrace{U_2^{\dagger} \sigma_i U_1^{\dagger}}_{V_i^{\dagger}}$$

∎

Examples of single qubit channels

1) Bit and phase flips

1.1) Bit flip

$$\Phi = p \, \mathbb{1} + (1-p) \, \Phi_1 .$$

$$M = (1-p) \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + p \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & 1-2p & \\ & & 1-2p \end{pmatrix}$$

1.2) Phase flip

$$\Phi = (1-p) \, \mathbb{1} + p \, \Phi_3 .$$

$$M = \begin{pmatrix} 1-2p & & \\ & 1-2p & \\ & & 1 \end{pmatrix}$$

1.3) Bit-phase flip

$$\Phi = (1-p) \, \mathbb{1} + p \, \Phi_2 .$$

$$M = \begin{pmatrix} 1-2p & & \\ & 1 & \\ & & 1-2p \end{pmatrix}$$

2) Depolarizing channel

$$\Phi(\rho) = p\,\frac{\mathbb{1}}{2} + (1-p)\,\rho$$

Completely dephasing channel $\Omega(\rho) = \mathbb{1}/2$

$$\Omega(\rho) = \frac{1}{4}\sum_{i=0}^{3} \Phi_i(\rho)$$

$$= \left(1 - \frac{3p}{4}\right)\mathbb{1} + \frac{p}{4}\sum_{i=1}^{3}\Phi_i$$

$$M = \left(1 - \frac{3p}{4}\right)\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \frac{p}{4}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= (1-p)\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

3) Phase damping

$$\Phi(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix}.$$

4) Amplitude damping

$$\Phi(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}.$$

$$\Phi(\mathbb{1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1-\gamma \end{pmatrix} = (1-\gamma/2)\,\mathbb{1} + \gamma/2\, Z$$

(not unital)

Exercise: find $[+ \vdots M]$ for (3) & (4).

$$\mathrm{Tr}_V\left(\mathrm{vec}\sqrt{P}\,(\mathrm{vec}\sqrt{P})^\dagger\right) = \sqrt{P}\,\sqrt{P}^\dagger$$

$$= P$$

Channel fidelity

The channel fidelity of $\Phi \in C(V,W)$ with respect to $P \in Pos(V)$ is defined by

$$F(\Phi, P) = F(|u\rangle\langle u|, \Phi \otimes \mathbb{1}_{L(V)}(|u\rangle\langle u|))$$

where $|u\rangle = vec(\sqrt{P})$. ($|u\rangle$ is a purification for P)

(Monotonicity of fidelity)

Pro: For $\Phi \in C(V,W)$ and $P, Q \in Pos(V)$ we have

$$F(P,Q) \leq F(\Phi(P), \Phi(Q)).$$

Proof: Let $A \in U(V, W \otimes U)$ be such that

$$\Phi(C) = Tr_U(A C A^\dagger).$$ (Stinespring representation)

Let $|u\rangle, |v\rangle \in V \otimes U'$ be purifications of P and Q such that

$$F(P,Q) = \langle u|v\rangle.$$ (Uhlmann thm)

Then

$$|\tilde{u}\rangle = A \otimes \mathbb{1}_{U'} |u\rangle$$

is a purification for $\Phi(P)$:

$$Tr_{U \otimes U'}(|\tilde{u}\rangle\langle\tilde{u}|) = Tr_{U \otimes U'}(A \otimes \mathbb{1}_{U'} |u\rangle\langle u| A^\dagger \otimes \mathbb{1}_{U'})$$

$$= Tr_U(A \, Tr_{U'}(|u\rangle\langle u|) A^\dagger)$$

$$= Tr_U(A P A^\dagger)$$

$$= \Phi(P).$$

Similarly $|\tilde{v}\rangle$ is a purification for $\Phi(Q)$.

Then

$$F(\Phi(P), \Phi(Q)) \geq \langle \tilde{u} | \tilde{v} \rangle$$
$$= \langle u | \underbrace{A^{+}A}_{\mathbb{1}_V} \otimes \mathbb{1}_{U'} \, v \rangle$$
$$= \langle u | v \rangle = F(P, Q). \quad \square$$

<u>Cor</u>: Let $\Phi \in C(V)$ and $P \in \text{Pos}(V)$.

For $|u\rangle \in V \otimes W$ and $Q \in \text{Pos}(V \otimes U)$

satisfying $P = \text{Tr}_W |u\rangle\langle u| = \text{Tr}_U Q$

we have

$$F(Q, \Phi \otimes \mathbb{1}_{L(U)}(Q)) \geq F(|u\rangle\langle u|, \Phi \otimes \mathbb{1}_{L(W)} |u\rangle\langle u|)$$

<u>Proof</u>: There exists $\Psi \in C(W, U)$ such that

$$\mathbb{1}_{L(V)} \otimes \Psi (|u\rangle\langle u|) = Q. \quad \text{(Proposition)}$$

By the monotonicity of fidelity

$$F(|u\rangle\langle u|, \Phi \otimes \mathbb{1}_{L(W)}(|u\rangle\langle u|))$$
$$\leq F(\mathbb{1}_{L(V)} \otimes \Psi(|u\rangle\langle u|), \Phi \otimes \Psi(|u\rangle\langle u|))$$
$$= F(Q, \Phi \otimes \mathbb{1}_{L(U)} Q) \quad \square$$

As a consequence

$$F(\Phi, P) = \min_{Q} \{ F(Q, \Phi \otimes \mathbb{1}_{L(U)}(Q)) : \text{Tr}_U Q = P \}$$

Writing

$$\Phi(A) = \sum_a A_a A A_a^+$$

we have

$$F(\Phi, P) = F\left(|u\rangle\langle u|,\ \Phi \otimes \mathbb{1}_{L(V)}(|u\rangle\langle u|)\right)$$

$$= \sqrt{\langle u, \Phi \otimes \mathbb{1}_{L(V)}(|u\rangle\langle u|)\, u\rangle}$$

$$= \sqrt{\sum_a \langle u, A_a \otimes \mathbb{1}_V |u\rangle\langle u| A_a^+ \otimes \mathbb{1}_V\, u\rangle}$$

$$= \sqrt{\sum_a |\underbrace{\langle u| A_a \otimes \mathbb{1}_V\, u\rangle}|^2}$$

$$\langle \mathrm{vec}\sqrt{P},\ A_a \otimes \mathbb{1}_V\, \mathrm{vec}\sqrt{P}\rangle$$

$$= \langle \sqrt{P},\ A_a \sqrt{P}\rangle$$ Exercise

$$= \sqrt{\sum_a |\langle P, A_a\rangle|^2}\ .$$