QUANTUM THEORY

We will do finite-dimensional quantum theory

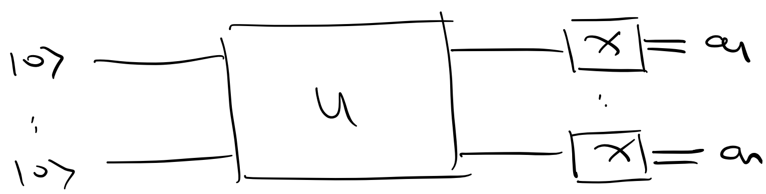
Axioms

Quantum theory consists of three components:

1) States
2) Transformations
3) Measurements

described by operators acting on $\mathcal{V}$.

In the circuit representation



$a_i \in \{0,1\}$

or equivalently

$$|0\cdots0\rangle \underset{\text{transform}}{\longmapsto} U|0\cdots0\rangle \underset{\text{measure}}{\longmapsto} p(a_1\cdots a_n)$$

$$\underbrace{p(a_1\cdots a_n)}_{\text{probability of observing } a_1\cdots a_n} = |\langle a_1\cdots a_n| U |0\cdots0\rangle|^2$$

$$U|0\cdots0\rangle = \sum_{a_1\cdots a_n} \langle a_1\cdots a_n|U|0\cdots0\rangle \, |a_1\cdots a_n\rangle$$

Observe that

$$1 = \| U |0\cdots 0\rangle \|^2 = \sum_{a_1\cdots a_n} |\langle a_1 \cdots a_n | U | 0\cdots 0\rangle|^2$$

That is,

$$\sum_{a_1\cdots a_n} \underbrace{p(a_1\cdots a_n)}_{\text{in } \mathbb{R}_{\geq 0}} = 1$$

Quantum state after the measurement:

If $p(a_1 \cdots a_n) > 0$ and $a_1 \cdots a_n$ is observed then the post-measurement state is

$$|a_1 \cdots a_n\rangle.$$

Ex: $|0\rangle - \boxed{H} - \boxed{\nearrow} = a$

$$\begin{aligned}
p(a) &= \langle a | H | 0 \rangle \\
&= \langle a | \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{\sqrt{2}} \left(\langle a | 0 \rangle + \langle a | 1 \rangle \right) \\
&= \begin{cases} 1/2 & a = 0 \\ 1/2 & a = 1 \end{cases}
\end{aligned}$$

Probabilities of outcomes are as in a coin toss:

If $a=0$ is observed then

post-measurement state is $|0\rangle$.

If $a=1$ is observed then

post-measurement state is $|1\rangle$.

We will study the components

1) States
2) Transformations
3) Measurements

individually from a more general point of view.

States

The state of a quantum system is specified by a density operator:

$$\rho \in \mathrm{Den}(V).$$

These are also called mixed states.

Quantum states of the form

$$\rho = |v\rangle\langle v| \quad , \|v\| = 1$$

are called pure states.

We will write $P(V)$ to denote the set of pure states.

Pro: The following sets are in one-to-one correspondence:

1) $P(V) = \{ |v\rangle\langle v| : v \in V \ , \|v\| = 1 \}$

2) $\mathrm{Proj}_1(V) = \{ \Pi \in \mathrm{Proj}(V) : \mathrm{Tr}\,\Pi = 1 \}$

3) $$\frac{\{ v \in V : \|v\| = 1 \}}{v \sim v' \text{ if } v' = \alpha v, \ \alpha \in U(\mathbb{C})}$$

4) $$\frac{\{ v \in V : v \neq 0 \}}{v \sim v' \text{ if } v' = \alpha v,\ \alpha \in \mathbb{C} - \{0\}}$$

Proof : (1) $\iff$ (2)

By definition $P(V) \subset \mathrm{Proj}_1(V)$:

$$\mathrm{Tr}\big(|v\rangle\langle v|\big) = \langle v | v \rangle = 1$$

Conversely, if $\Pi \in \mathrm{Proj}_1(V)$ let $v \in V_\Pi$: $\|v\| = 1$. Then

$$\Pi = |v\rangle\langle v| .$$

HW : $\dim V_\Pi = 1$.

(2) $\iff$ (4) : HW.

(3) $\iff$ (1)

Define a function

$$\frac{\{ v : \|v\| = 1 \}}{v \sim \alpha v,\ \alpha \in U(\mathbb{C})} \xrightarrow{\cong} P(V)$$

$$|v\rangle \longmapsto |v\rangle\langle v|$$

This function is surjective.

The function is injective :

$$v \sim \alpha v \implies \alpha |v\rangle\langle v| \bar{\alpha} = |\alpha|^2 |v\rangle\langle v| = |v\rangle\langle v|$$

$\blacksquare$

Single qubit states

$$A^2 = \frac{1}{2}\left(\frac{\alpha_0^2 + |\vec{\alpha}|^2}{2} \mathbb{1} + \alpha_0 \sum_{i=1}^{3} \alpha_i \sigma_i\right)$$

Let $V = \mathbb{C}^2$.

First approach: $P(V) \subset \mathrm{Herm}(V)$

Recall that $\{\mathbb{1}, X, Y, Z\}$ is an orthonormal basis for $\mathrm{Herm}(V)$:

$$A = \frac{1}{2}(\alpha_0 \mathbb{1} + \alpha_1 X + \alpha_2 Y + \alpha_3 Z).$$

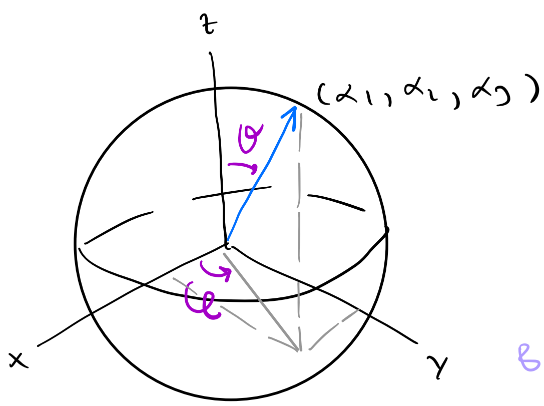
We have $A \in P(V) \iff$

$$\mathrm{Tr}(A) = 1 \quad \& \quad A^2 = A.$$

$$\mathrm{Tr}(A) = \frac{1}{2} \alpha_0 \mathrm{Tr}(\mathbb{1}) = \boxed{\alpha_0 = 1}$$

$$A^2 = \frac{1}{4}\Big((1 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)\, \mathbb{1} \; 2(\alpha_1 X + \alpha_2 Y + \alpha_3 Z)\Big) = A$$

$$\iff \boxed{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1.}$$



Bloch sphere

Second approach: $P(V) = \{ |v\rangle : \|v\| = 1 \} / \sim$

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$= r_0 e^{i\gamma_0}|0\rangle + r_1 e^{i\gamma_1}|1\rangle$$

$$= e^{i\gamma_0}\left(r_0|0\rangle + r_1 e^{i(\gamma_1-\gamma_0)}|1\rangle \right) \qquad r_0^2 + r_1^2 = 1$$

$$\sim r_0|0\rangle + r_1 e^{i\underbrace{(\gamma_1-\gamma_0)}_{\varphi}}|1\rangle$$

$$= \cos\frac{\vartheta}{2}|0\rangle + \sin\frac{\vartheta}{2} e^{i\varphi}|1\rangle$$

$$0 \le \vartheta < \pi, \qquad 0 \le \varphi < 2\pi .$$

Other important operators

$$\begin{tikzcd}
 & \mathrm{Herm}(\mathbb{C}^2) \arrow[d, dash] & \\
 & \mathrm{Pos}(\mathbb{C}^2) \arrow[d, dash] \arrow[dr, dash] & \\
 & \mathrm{Proj}(\mathbb{C}^2) \arrow[dr, dash] & \mathrm{Den}(\mathbb{C}^2) \arrow[d, dash] \\
 & & P(\mathbb{C}^2)
\end{tikzcd}$$

Hermitian operators:

$$A = \begin{pmatrix} r_1 & s_1 + i s_2 \\ s_1 - i s_2 & r_2 \end{pmatrix} \qquad r_i, s_i \in \mathbb{R}$$

HW eig. val: $\dfrac{\alpha_0 \pm |\hat{\alpha}|}{2}$

$$A = \frac{1}{2}\left(\sum_{i=0}^{3} \underbrace{\langle \sigma_i, A \rangle}_{\alpha_i} \sigma_i \right)$$

$$\langle \mathbb{1}, A\rangle = \mathrm{Tr}(\mathbb{1} A) = \mathrm{Tr}\, A = r_1 + r_2$$

$$\langle X, A\rangle = \mathrm{Tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A\right)$$

$$= \mathrm{Tr}\begin{pmatrix} s_1 - i s_2 & r_2 \\ r_1 & s_1 + i s_2 \end{pmatrix} = 2 s_1$$

$$\langle Z, A\rangle = \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A\right)$$

$$= \mathrm{Tr}\begin{pmatrix} r_1 & s_1 + i s_2 \\ -s_1 + i s_2 & -r_2 \end{pmatrix} = r_1 - r_2$$

$$\langle Y, A\rangle = \mathrm{Tr}\left(i X Z A \right)$$

$$= i\, \mathrm{Tr}\begin{pmatrix} -s_1 + i s_2 & -r_2 \\ r_1 & s_1 + i s_2 \end{pmatrix} = -2 s_2$$

$$A = \frac{1}{2}\Big(\underbrace{(r_1 + r_2)}_{\alpha_0}\, \mathbb{1} + \underbrace{2 s_1}_{\alpha_1}\, X + \underbrace{(-2 s_2)}_{\alpha_2}\, Y + \underbrace{(r_1 - r_2)}_{\alpha_3}\, Z \Big)$$

$$\mathrm{Tr}(A) = \alpha_0 = \lambda_1 + \lambda_2$$

λ_i : eigenvalues of A.

$$\mathrm{Det}(A) = r_1 r_2 - (s_1^2 + s_2^2)$$

$$= \frac{1}{4}\left(\alpha_0^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 \right) = \lambda_1 \lambda_2$$

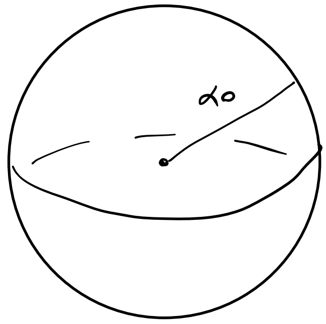
Eigenvalues are the solutions of

$$\lambda^2 - \mathrm{Tr}\, A\, \lambda + \mathrm{Det}\, A = 0$$

$\mathrm{Pos}(\mathbb{C}^2)$: λ_1 & $\lambda_2 \geq 0$ $\iff$ $\mathrm{Tr}\, A \geq 0$

$\mathrm{Det}\, A \geq 0$

$\iff$ $\alpha_0 \geq 0$ & $\alpha_0^2 \geq \alpha_1^2 + \alpha_2^2 + \alpha_3^2$.



ball of radius $\leq \alpha_0$

$\mathrm{Den}(\mathbb{C}^2)$: Ball of radius $= 1$:

$\alpha_0 = 1$ & $1 \geq \alpha_1^2 + \alpha_2^2 + \alpha_3^2$

$P(\mathbb{C}^2)$: $\mathrm{Tr}\, A = 1$, $\mathrm{Det}\, A = 0$ $\iff$

$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. (Bloch sphere)

<u>Ex</u> : $|0\rangle\langle 0| = \frac{1}{2} \sum_i \mathrm{Tr}(\sigma_i\, |0\rangle\langle 0|)\, \sigma_i$

$= \frac{1}{2} \sum_i \langle 0|\sigma_i|0\rangle\, \sigma_i$

$= \frac{1}{2} (\mathbb{1} + Z)$

$|1\rangle\langle 1| = \frac{1}{2} (\mathbb{1} - Z)$

$|\pm\rangle\langle \pm| = \frac{1}{2} (\mathbb{1} \pm X)$

$|\pm i\rangle\langle \pm i| = \frac{1}{2} (\mathbb{1} \pm Y)$

Transformations

Time evolution of a quantum state is described by a unitary operator

$$\underbrace{|v'\rangle}_{\text{state at } t_1} = U \underbrace{|v\rangle}_{\text{state at } t_0} .$$

The exponential map:

$$\exp : \mathrm{Herm}(V) \longrightarrow U(V)$$
$$A \longmapsto e^{-iA}$$

Spectral decomposition:

$$A = \sum_i \lambda_i v_i v_i^{+} \qquad \lambda_i \in \mathbb{R}$$

$$e^{-iA} = \sum_i \underbrace{e^{-i\lambda_i}}_{\text{in } U(\mathbb{C})} v_i v_i^{+}$$

Therefore exp is surjective.

The action

$U(V)$ acts on $\mathrm{Herm}(V)$ by conjugation:

$$B \mapsto U B U^{+}, \quad B \in \mathrm{Herm}(V).$$

Observations

1) $B = \alpha \mathbb{1}$, $\alpha \in \mathbb{R}$, is fixed:

$$U \alpha \mathbb{1} U^{+} = \mathbb{1} \qquad \forall\, U \in U(V).$$

2) $U = e^{-iA}$ & $A = \beta \mathbb{1}$, $\beta \in \mathbb{R}$, then

$$U B U^{+} = B, \qquad \forall\, B \in \mathrm{Herm}(V).$$

To understand the action of $U(V)$ we restrict to

$$U = e^{-iA} \quad \text{where} \quad \langle \mathbb{1}, A \rangle = 0.$$

Moreover, we will fix $\| A \|$ some constant c. Then

$$U = e^{-itA} \qquad \| A \| = c$$

$$t \in \mathbb{R}.$$

Schrödinger equation (time evolution):

Let $U = e^{-itA}$, $t \in \mathbb{R}$:

$$|v'\rangle = e^{-itA} |v\rangle$$

Then

$$\frac{d}{dt}\Big(\underbrace{e^{-itA} |v\rangle}_{|v'\rangle}\Big) = -iA \underbrace{e^{-itA} |v\rangle}_{|v'\rangle}.$$

This gives the Schrödinger eq:

$$\frac{d}{dt} |v'\rangle = -iA |v'\rangle$$

(Time evolution)

Here $t \in \mathbb{R}$ represents time.

Single qubit rotations

Let $V = \mathbb{C}^2$.

$\{ \mathbb{1}, X, Y, Z \}$ orthonormal basis for $\mathrm{Herm}(V)$

Using exp map any Hermitian can be written:

$$e^{-it \underbrace{\frac{1}{2}\sum_{i=0}^{3} \alpha_i \sigma_i}_{A}} = \underbrace{e^{-it \frac{\alpha_0}{2} \mathbb{1}}}_{e^{i\alpha}\mathbb{1}} \; e^{-it \frac{1}{2} \sum_{i=1}^{3} \alpha_i \sigma_i}$$

For the action on $\mathrm{Herm}(\mathbb{C}^2)$ we will assume $\alpha_0 = 0$:

$$A = \frac{1}{2} (\alpha_1 X + \alpha_2 Y + \alpha_3 Z)$$

Notation:
$$\hat{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$$
$$\hat{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$
$$\hat{\alpha} \cdot \hat{\sigma} = \sum_{i=1}^{3} \alpha_i \sigma_i .$$

With this notation

$$e^{-itA} = e^{-it \hat{\alpha} \cdot \hat{\sigma} / 2}$$

We will consider A with

$$|\hat{\alpha}| = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{1/2} = 1,$$

that is, $\| A \| = |\hat{\alpha}| / \sqrt{2} = 1/\sqrt{2}$.

Z-rotation: $\hat{\alpha} = (0,0,1) \Rightarrow \hat{\alpha}\cdot\hat{\sigma} = Z$:

$$e^{-itZ/2} = \cos(t/2)\,\mathbb{1} - i\sin(t/2)\,Z$$

$$= \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix}$$

Action on $\mathrm{Span}_{\mathbb{R}}\{X, Y, Z\}$

$$X \longmapsto e^{-itZ/2}\, X\, e^{itZ/2}$$

$$= \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}}_{\begin{pmatrix} 0 & e^{-it/2} \\ e^{it/2} & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} 0 & e^{-it} \\ e^{it} & 0 \end{pmatrix} = \cos t\, X + \sin t\, Y$$

$$Z \longmapsto e^{-itZ/2}\, Z\, e^{itZ/2} = Z$$

$$Y \longmapsto e^{-itZ/2}\,(iXZ)\,e^{itZ/2}$$

$$= i\, e^{-itZ/2}\, X\, e^{itZ/2}\, Z$$

$$= \cos t\, Y - \sin t\, X$$

As a matrix acting on $\mathbb{R}^3$:

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For example

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\, e^{-i\pi Z/2}$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = e^{i\pi/4}\, e^{-i\pi Z/4}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/8}\, e^{-i\pi Z/8}$$

Y-rotation:

$$e^{-itY/2} \longmapsto \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}$$

X-rotation:

$$e^{-itX/2} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

<u>Lem</u>:

$$e^{-it\hat{\alpha}\cdot\hat{\sigma}/2} = \cos\left(\frac{t}{2}\right)\mathbb{1} - i\sin\left(\frac{t}{2}\right)\hat{\alpha}\cdot\hat{\sigma}$$

<u>Proof</u>: If $B^2 = \mathbb{1}$ then

$$e^{-itB/2} = \underbrace{e^{-it/2}}_{\cos t/2 - i\sin t/2}\Pi^0_B + \underbrace{e^{it/2}}_{\cos t/2 + i\sin t/2}\Pi^1_B$$

$$= \cos t/2\,(\Pi^0_B + \Pi^1_B) - i\sin t/2\,(\Pi^0_B - \Pi^1_B)$$

$$= \cos t/2\,\mathbb{1} - i\sin t/2\,B$$

We have

$$(\hat{\alpha}\cdot\hat{\sigma})^2 = \Big(\sum_{i=1}^{3}\alpha_i\sigma_i\Big)^2$$

$$= \sum_{i,j}\alpha_i\alpha_j\,\sigma_i\sigma_j$$

$$= \sum_i \alpha_i^2\,\mathbb{1} + \cancel{\sum_{i<j}\alpha_i\alpha_j\,\sigma_i\sigma_j} + \cancel{\sum_{j<i}\alpha_i\alpha_j}\,\underbrace{\sigma_i\sigma_j}_{-\sigma_j\sigma_i}$$

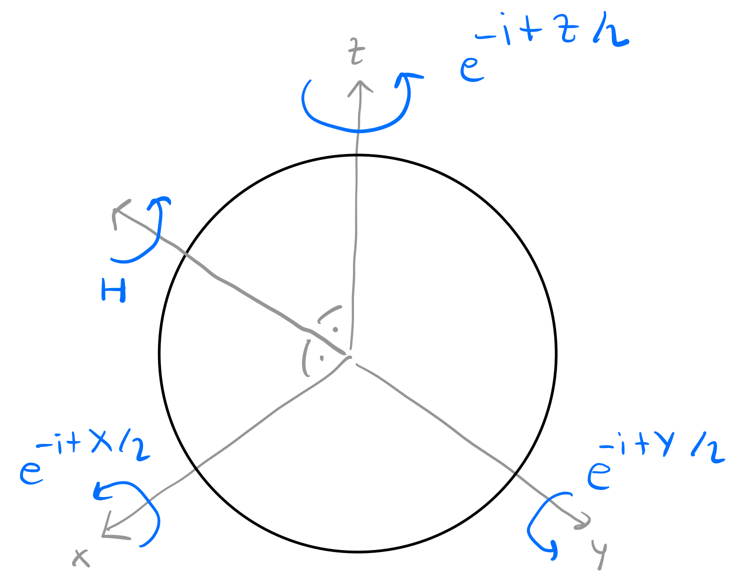
$$= \underbrace{|\alpha|^2}_{1}\,\mathbb{1} = \mathbb{1}.$$

Combining these two observations finishes the proof. $\blacksquare$

Theorem : $R_{\hat{\alpha}}(t)$ rotation by angle t about the $\hat{\alpha}$ axis.

Hadamard :

$$\begin{array}{ccc} X & & HXH = Z \\ Y & \longmapsto & HYH = -Y \\ Z & & HZH = X \end{array}$$



HW : $H = R_{\hat{\alpha}}(\pi)$

$\hat{\alpha} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

Measurements

A quantum measurement is specified by a function
$$M : \Sigma \longrightarrow L(V) \quad \text{such that}$$
$$\sum_{a \in \Sigma} M_a^\dagger M_a = \mathbb{1} .$$

Σ is interpreted as the set of outcomes.
If $\rho \in \mathrm{Den}(V)$ is the state of the system then
$$p(a) = \overbrace{\mathrm{Tr}\left(M_a \rho M_a^\dagger\right)}^{\mathrm{Tr}(\rho M_a^\dagger M_a)}$$
is the probability of obtaining outcome $a \in \Sigma$ (Born rule).
If $a \in \Sigma$ is observed then the state of of the system after the measurement is
$$\frac{M_a \rho M_a^\dagger}{\mathrm{Tr}\left(M_a \rho M_a^\dagger\right)} .$$

If $M_a \in \mathrm{Proj}(V)$ $\forall\, a \in \Sigma$ then M is called a projective measurement.

If $\Sigma \subset \mathbb{R}$ then a projective measurement

$$M : \Sigma \longrightarrow \mathrm{Proj}(V)$$

can be assembled into a Hermitian op:

$$A = \sum_{\lambda \in \Sigma} \lambda \, \Pi_\lambda . \quad \text{(observable)}$$

Conversely, $A \in \mathrm{Herm}(V)$ gives a projective meas. by the spectral decomposition theorem.

Lem: Let $\Pi : \Sigma \longrightarrow \mathrm{Proj}(V)$ be a projective measurement. Then

$$\langle \Pi_a , \Pi_b \rangle = 0 \qquad \forall \text{ distinct } a, b \in \Sigma$$

Cor: $\Pi_a \Pi_b = 0 \quad a \neq b$

Proof: Since $\sum_a \Pi_a = \mathbb{1}$, its square gives

$$\underbrace{\sum_{a,b} \Pi_a \Pi_b}_{\sum_a \Pi_a + \sum_{a \neq b} \Pi_a \Pi_b} = \mathbb{1} = \sum_a \Pi_a$$

Thus $\sum_{a \neq b} \Pi_a \Pi_b = 0$. Taking trace

$$\mathrm{Tr}(\Pi_a \Pi_a \Pi_b \Pi_b) = \mathrm{Tr}(\Pi_a \Pi_b \Pi_b \Pi_a)$$

$$\sum_{a \neq b} \underbrace{\mathrm{Tr}(\Pi_a \Pi_b)}_{\langle \Pi_a, \Pi_b \rangle \geq 0 \text{ since } \Pi_a, \Pi_b \text{ positive.}} = \mathrm{Tr}(0) = 0.$$

$$= \langle \Pi_b \Pi_a , \Pi_b \Pi_a \rangle \geq 0$$

Therefore $\langle \Pi_a , \Pi_b \rangle = 0$ for $a \neq b$. $\square$

A positive operator valued measure (POVM) is a function

$$P : \Sigma \to \mathrm{Pos}(V) \quad \text{s.t.} \quad \sum_{a \in \Sigma} P_a = \mathbb{1}.$$

Every quantum measurement M gives a POVM:

$$P_a = M_a^+ M_a.$$

Ex: Let $\{v_a\}_{a \in \Sigma}$ be an orthonormal basis for V.

Then $\Pi : \Sigma \to L(V)$

$$\Pi_a = v_a v_a^+$$

is a projective measurement.

If ρ is pure, i.e. of the form ww^+, then

$$\begin{aligned} p(a) &= \mathrm{Tr}(\Pi_a \rho \Pi_a) \\ &= \mathrm{Tr}(\Pi_a^+ \Pi_a \rho) \\ &= \mathrm{Tr}(\Pi_a \rho) \\ &= \mathrm{Tr}(v_a v_a^+ w w^+) \\ &= |\langle v_a, w \rangle|^2 \end{aligned}$$

$$\left. \begin{aligned} M_a^+ &= M_a \\ M_a^2 &= M_a \end{aligned} \right\}$$

and the post-measurement state:

$$\rho' = \frac{\Pi_a \rho \Pi_a}{p(a)}$$

$$= \frac{v_a v_a^\dagger w w^\dagger v_a v_a^\dagger}{p(a)}$$

$$= \frac{|\langle v_a, w\rangle|^2}{p(a)} v_a v_a^\dagger$$

$$= v_a v_a^\dagger$$

Alternatively $\rho' = u u^\dagger$ where

$$u = \frac{\langle v_a, w\rangle}{\sqrt{p(a)}} v_a$$

$$= \frac{\Pi_a w}{\|\Pi_a w\|}.$$

Single qubit measurements

Let $\Pi : \Sigma \longrightarrow \mathrm{Proj}(\mathbb{C}^2)$ be a projective measurement.

Then $|\Sigma| \leq 2$:

- $\Sigma = \{a\}$ then $\Pi_a = \mathbb{1}$,
- $\Sigma = \{a, b\}$ then

$$\langle \Pi_a , \Pi_b \rangle = 0 .$$

Let us identify Σ with $\{0, 1\}$.

We will consider projective measurements

$$\Pi : \Sigma \longrightarrow \mathrm{Proj}(V).$$

These are precisely those $A \in \mathrm{Herm}(V)$ with eigenvalues ± 1.

Lem: $A \in \mathrm{Herm}(V)$ has eig. val ± 1

$\Leftrightarrow$ $A = p_1 X + p_2 Y + p_3 Z$, $|p| = 1$.

Proof: $(\Rightarrow)$

A has eigval. ± 1:

$$A = U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^{+}$$

Then $A^2 = (U Z U^{+})^2 = \mathbb{1}$.

Writing

$$A = \frac{1}{2} \sum_{i=0}^{3} \alpha_i \sigma_i = \frac{1}{2} (\alpha_0 \mathbb{1} + \hat{\alpha} \cdot \hat{\sigma})$$

we find that

$$A^2 = \frac{\alpha_0^2 + |\hat{\alpha}|^2}{4} \mathbb{1} + \frac{\alpha_0}{2} \hat{\alpha} \cdot \hat{\sigma} = \mathbb{1}$$

$$\iff \alpha_0 = 0 \quad \& \quad |\alpha| = 2.$$

Therefore $A = \hat{p} \cdot \hat{\sigma}$ for some $|p| = 1$.

$(\Leftarrow)$ $A = p_1 X + p_2 Y + p_3 Z$ with $|p| = 1$ then

$$\mathrm{Tr}\, A = 2p_0 = 0$$

$$\mathrm{Det}\, A = p_0^2 - |\hat{p}|^2 = -1$$

$\iff$ A has eigenvalues ± 1. $\boxtimes$

Let $\rho = vv^\dagger = \frac{1}{2}(\mathbb{1} + \hat{\alpha} \cdot \hat{\sigma})$

$A = \hat{p} \cdot \hat{\sigma}$ where $|\alpha| = |p| = 1$.

Born rule

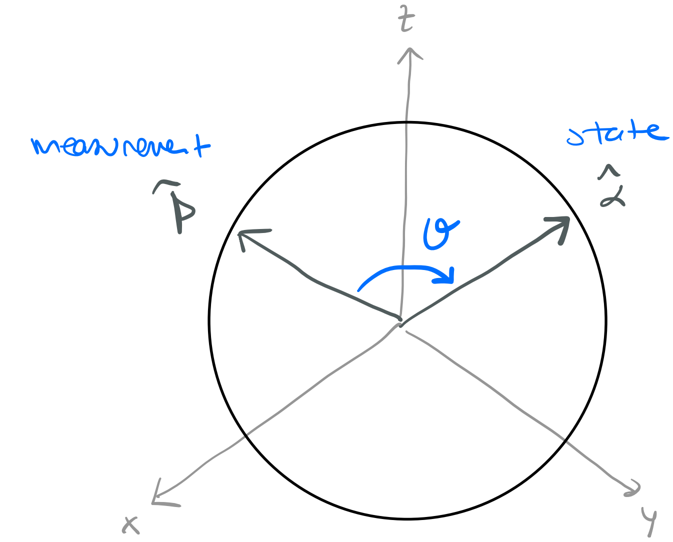
$$p(a) = \mathrm{Tr}(\rho \Pi_a)$$

$$= \langle \rho, \Pi_a \rangle$$

$$= \left\langle \frac{1}{2}(\mathbb{1} + \hat{\alpha} \cdot \hat{\sigma}), \frac{1}{2}(\mathbb{1} + (-1)^a \hat{p} \cdot \hat{\sigma}) \right\rangle$$

$$= \frac{1}{2}\Big(1 + (-1)^a \underbrace{\hat{\alpha} \cdot \hat{p}}_{|\alpha|\,|p| \cos\theta \,=\, \cos\theta}\Big)$$

Note $\Pi_a = \frac{1}{2}(\mathbb{1} + (-1)^a A)$.



- $p(0) = \dfrac{1 + \cos\vartheta}{2}$

  post-meas. state

  +1 eigvec.

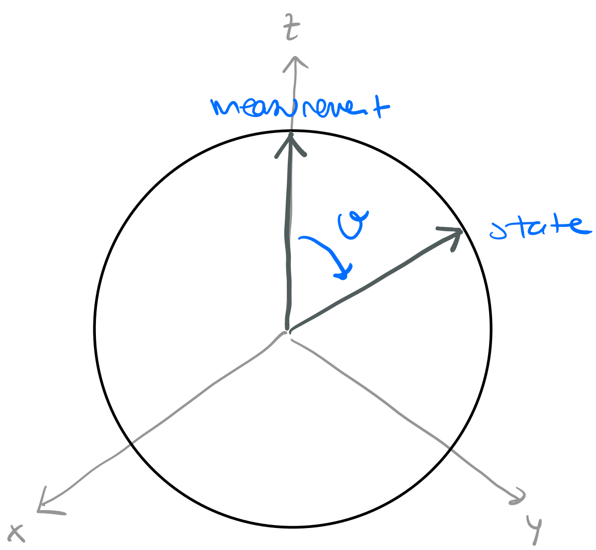
- $p(1) = \dfrac{1 - \cos\vartheta}{2}$

  post-meas. state

  −1 eigvec.

Quantum circuit

$|0\rangle$ —— $\boxed{U}$ —— $\boxed{x}$ $=$ a



Composing systems

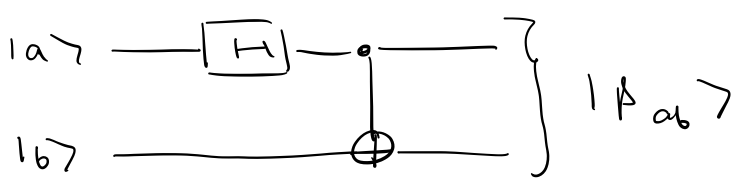
The Hilbert space of a composite quantum system, consisting of two quantum systems with Hilbert spaces V and W, is the tensor product
$$V \otimes W.$$

We have

1) states : $\rho \in \mathrm{Den}(V \otimes W)$
2) transformations : $U \in U(V \otimes W)$
3) measurements : $M : \Sigma \longrightarrow L(V \otimes W)$

A state $\rho \in \mathrm{Den}(V \otimes W)$ is called a product state if
$$\rho = \rho_1 \otimes \rho_2$$
for some $\rho_1 \in \mathrm{Den}(V)$ and $\rho_2 \in \mathrm{Den}(W)$.
Otherwise, ρ is called entangled.

Ex :



$$|\beta_{ab}\rangle = \mathrm{CNOT}\; H \otimes \mathbb{1}\; |a\rangle \otimes |b\rangle$$
$$= \mathrm{CNOT}\; \frac{|0\rangle + (-1)^a |1\rangle}{\sqrt{2}} \otimes |b\rangle$$

$$= \frac{1}{\sqrt{2}} \text{CNOT} \left(|0\rangle \otimes |b\rangle + (-1) \; |1\rangle \otimes |b\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(\text{CNOT} \; |0\rangle \otimes |b\rangle + (-1)^a \; \text{CNOT} \; |1\rangle \otimes |b\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(|0 b\rangle + (-1)^a \; |1 \, (b+1)\rangle \right)$$

$$= Z^a \otimes X^b \; |\beta_{00}\rangle .$$

Assume $\quad |\beta_{ab}\rangle = |v\rangle \otimes |w\rangle \quad$ for

some $\quad v, w \in P(\mathbb{C}^2)$.

Then

$$|\beta_{00}\rangle = Z^a |v\rangle \otimes X^b |w\rangle .$$

That is, we can take $a = b = 0$!

$$|\beta_{00}\rangle = \left(\alpha_0 |0\rangle + \alpha_1 |1\rangle \right) \otimes \left(\beta_0 |0\rangle + \beta_1 |1\rangle \right)$$

$$= \underbrace{\alpha_0 \beta_0}_{\frac{1}{\sqrt{2}}} |00\rangle + \underbrace{\alpha_0 \beta_1}_{0} |01\rangle + \underbrace{\alpha_1 \beta_0}_{0} |10\rangle + \underbrace{\alpha_1 \beta_1}_{\frac{1}{\sqrt{2}}} |11\rangle$$

There exist no such $\alpha_i, \beta_i \in \mathbb{C}$, thus

$|\beta_{ab}\rangle$ is entangled.

Teleportation

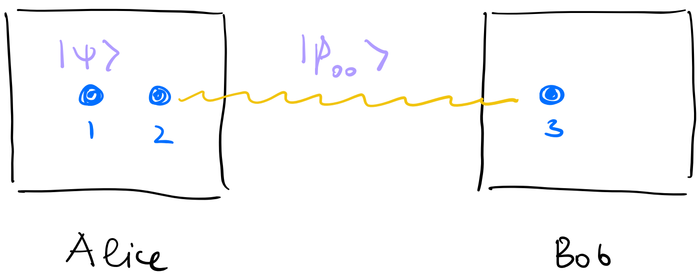
Consider two parties: Alice and Bob.

Alice owns two qubits: V_1, V_2.

Bob owns one qubit: V_3.

The initial state of the system:

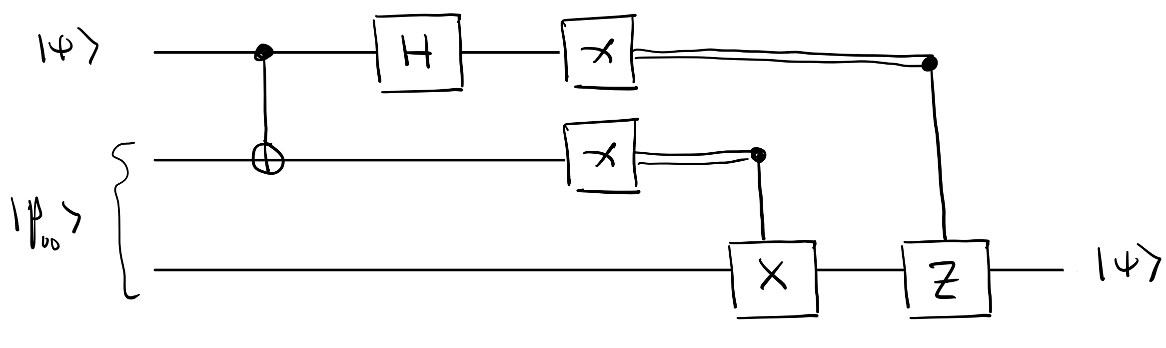
$$|\psi\rangle \otimes |\beta_{00}\rangle.$$



The goal is to send the quantum state $|\psi\rangle$ that Alice has to Bob:

$$|\times\rangle \otimes |\psi\rangle.$$

The quantum circuit:



Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$.

Then

$$(H \otimes \mathbb{1})\, CNOT_{12} |\psi\rangle |\beta_{00}\rangle =$$

$$(H\otimes \mathbb{1})\, CNOT_{12} \frac{1}{\sqrt{2}} \left(\alpha |0\rangle \left(|00\rangle + |11\rangle\right) + \beta |1\rangle \left(|00\rangle + |11\rangle\right)\right)$$

$$= \frac{1}{\sqrt{2}} (H \otimes \mathbb{1}) \left(\alpha |0\rangle \left(|00\rangle + |11\rangle\right) + \beta |1\rangle \left(|10\rangle + |01\rangle\right)\right)$$

$$= \frac{1}{\sqrt{2}} \left(\alpha |+\rangle \left(|00\rangle + |11\rangle\right) + \beta |-\rangle \left(|10\rangle + |01\rangle\right)\right)$$

$$= \frac{1}{2} \Big(\alpha \left(|0\rangle + |1\rangle\right)\left(|00\rangle + |11\rangle\right)$$

$$+ \beta \left(|0\rangle - |1\rangle\right)\left(|10\rangle + |01\rangle\right)\Big)$$

$$= \frac{1}{2} \Big(|00\rangle \left(\alpha|0\rangle + \beta|1\rangle\right) + |01\rangle \left(\alpha|1\rangle + \beta|0\rangle\right)$$

$$+ |10\rangle \left(\alpha|0\rangle - \beta|1\rangle\right) + |11\rangle \left(\alpha|1\rangle - \beta|0\rangle\right)\Big)$$

Alice performs Z – meas :

$$\Pi : \{00, 01, 10, 11\} \longrightarrow \mathrm{Proj}(V_1 \otimes V_2)$$

$$\Pi_{ab} = |a\rangle\langle a| \otimes |b\rangle\langle b|$$

$$= |ab\rangle \langle ab| .$$

After the measurement:

ab	post-meas. state	Correction $\mathbb{1} \otimes Z^a X^b$
00	$\lvert 00\rangle \left(\alpha \lvert 0\rangle + \beta \lvert 1\rangle \right)$	$\mathbb{1} \otimes \mathbb{1}$
01	$\lvert 01\rangle \left(\alpha \lvert 1\rangle + \beta \lvert 0\rangle \right)$	$\mathbb{1} \otimes X$
10	$\lvert 10\rangle \left(\alpha \lvert 0\rangle - \beta \lvert 1\rangle \right)$	$\mathbb{1} \otimes Z$
11	$\lvert 11\rangle \left(\alpha \lvert 1\rangle - \beta \lvert 0\rangle \right)$	$\mathbb{1} \otimes Z X$

Bob's state

Superdense coding

Consider two parties: Alice and Bob.

Alice owns one qubit: V_1.

Bob owns one qubit: V_2.

The initial state of the system:

$$|\beta_{00}\rangle.$$

Alice Bob

The goal is to send two bits ab from Alice to Bob:

1) Alice applies $Z^a X^b$ to V_1.
2) Alice sends her qubit to Bob.
3) Bob measures in the Bell basis:

$$\{ |\beta_{ab}\rangle : a, b \in \{0,1\} \}$$

Bob observes cd with probability:

$$p(cd) = |\langle \beta_{cd} | \underbrace{Z^a X^b \otimes \mathbb{1} | \beta_{00}\rangle}| ^2$$

$$\frac{1}{\sqrt{2}} Z^a X^b \otimes \mathbb{1} (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \left((-1)^{ab} |b\,0\rangle + (-1)^{a(b+1)} |(1+b)\,1\rangle \right)$$

$$= \frac{(-1)^{ab}}{\sqrt{2}} \left(|b\,0\rangle + (-1)^a |(1+b)\,1\rangle \right)$$

$$\sim |\beta_{..}\rangle$$

$|ab$

$$= |\langle \beta_{cd} | \beta_{ab} \rangle|^2$$

$$= \delta_{cd,ab} \, .$$

Operational meaning:

1) Quantum teleportation: 1 qubit is transmitted using 2 bits and an entangled state.

2) Superdense coding: 2 bits are transmitted using 1 qubit and an entangled state.

Density operators

We begin with a characterization of positive operators.

Pro: Let $P \in L(V)$.
The following are equivalent:

1) $\langle v, Pv \rangle \in \mathbb{R}_{\geq 0}$ $\forall v \in V$.
2) $P \in \mathrm{Herm}(V)$ and its eigenvalues in $\mathbb{R}_{\geq 0}$.
3) $P = A^{+} A$ for some $A \in L(V)$.
4) $\langle Q, P \rangle \in \mathbb{R}_{\geq 0}$ $\forall Q \in \mathrm{Pos}(V)$.

Proof: (HW)

$(1 \Rightarrow 2)$ P is Hermitian:

$$\langle P^{+} v, v \rangle = \langle v, Pv \rangle$$
$$= \overline{\langle Pv, v \rangle}$$
$$= \langle Pv, v \rangle$$

i.e. $\langle (P^{+} - P) v, v \rangle = 0$ $\forall v \in V$.

Therefore $P^{+} - P = 0$. (HW: Use

$$\langle v, Au \rangle = \frac{\langle u+v, A(u+v)\rangle - \langle u-v, A(u-v)\rangle}{4} + i \frac{\langle u+iv, A(u+iv)\rangle - \langle u-iv, A(u-iv)\rangle}{4}$$

).

$\Rightarrow P \in \mathrm{Herm}(V)$.

By spectral decomposition:

$$P = \sum_i \lambda_i v_i v_i^+ \quad , \quad \lambda_i \in \mathbb{R}.$$

Since $\langle v, Pv \rangle \in \mathbb{R}_{\geq 0}$ we have

$$\lambda_i = \langle v_i, P v_i \rangle \in \mathbb{R}_{\geq 0}$$

($2 \Rightarrow 3$) By spectral decomposition

$$\begin{aligned} P &= \sum_i \lambda_i v_i v_i^+ \\ &= \Big(\sum_i \sqrt{\lambda_i}\, v_i v_i^+\Big)\Big(\sum_i \sqrt{\lambda_i}\, v_i v_i^+\Big) \\ &= \underbrace{\sqrt{P}}_{(\sqrt{P})^+ = \sqrt{P}} \cdot \sqrt{P} \end{aligned}$$

($3 \Rightarrow 4$) We have

$$\begin{aligned} \langle Q, P \rangle &= \langle B^+B, A^+A \rangle \\ &= \mathrm{Tr}(B^+B\, A^+A) \\ &= \mathrm{Tr}(BA^+AB^+) \\ &= \langle AB^+, AB^+ \rangle \in \mathbb{R}_{\geq 0}. \end{aligned}$$

($4 \Rightarrow 1$) Apply to $Q = vv^+$:

$$\begin{aligned} \langle v, Pv \rangle &= \mathrm{Tr}(v^+ P v) \\ &= \mathrm{Tr}(vv^+ P) \\ &= \langle vv^+, P \rangle \geq 0 \qquad \blacksquare \end{aligned}$$

Characterization of density operators

For a subset $X \subseteq \mathbb{R}^n$ we will write

$$\mathrm{Conv}(X) = \Big\{ \sum_{x \in X} p_x \, x \; : \; p_x \geq 0 \, , \; p_x \neq 0 \text{ for finitely many } x \in X, \; \sum_x p_x = 1 \Big\}$$

for the convex hull of X.

Theorem: $\mathrm{Den}(V) = \mathrm{Conv}(P(V))$

proof: $(\Leftarrow)$

Let $A = \sum_i p_i \, v_i v_i^{+}$.

Positivity: $\langle v, Av \rangle \geq 0, \; \forall v \in V$:

$$\langle v, \sum_i p_i \, v_i v_i^{+} v \rangle = \sum_i p_i \underbrace{\langle v, v_i \underbrace{v_i^{+} v}_{\langle v_i, v \rangle} \rangle}_{|\langle v_i, v \rangle|^2}$$

$$\geq 0$$

Trace: $\mathrm{Tr}(A) = 1$:

$$\mathrm{Tr}\Big(\sum_i p_i \, v_i v_i^{+}\Big) = \sum_i p_i \overbrace{\mathrm{Tr}(v_i v_i^{+})}^{\langle v_i, v_i \rangle = 1}$$

$$= \sum_i p_i = 1$$

($\Rightarrow$) Let $\rho \in \mathrm{Den}(V)$.

Spectral decomposition gives:

$$\rho = \sum_j \lambda_j v_j v_j^+ \qquad \lambda_j \in \mathbb{R}_{\geq 0}$$

$$\mathrm{Tr}(\rho) = 1 \implies \sum_j \lambda_j = 1 .$$

Therefore $\rho \in \mathrm{Conv}(P(V))$. $\boxtimes$

Characterization of pure states:

1) $\mathrm{Tr}(\rho^2) \leq 1 \qquad \forall \rho \in \mathrm{Den}(V)$.

2) $\mathrm{Tr}(\rho^2) = 1 \iff \underbrace{\rho \in P(V)}_{\mathrm{Tr}(\rho)=1 \ \& \ \rho^2 = \rho}$.

proof: 1) Spectral dec.:

$$\rho = \sum_j \lambda_j v_j v_j^+$$

Then

$$\rho^2 = \sum_j \lambda_j^2 v_j v_j^+$$

and $\mathrm{Tr}(\rho^2) = \sum_j \lambda_j^2 \leq \sum_j \lambda_j = 1$.

2) Restatement:

$$\sum_j \lambda_j^2 = 1 \iff \lambda_j \in \{0,1\} \ \forall j \ \& \ \sum_j \lambda_j = 1 .$$

To see this

($\Leftarrow$) Clear.

($\Rightarrow$) If $0 < \lambda_j < 1$ for some j then

$$\sum_j \lambda_j^2 < \sum_j \lambda_j = 1. \qquad \boxtimes$$

<u>Ex</u> Let $\rho \in D(\mathbb{C}^2)$:

$$\rho = \frac{1}{2}\left(\sigma_0 + \sum_{i=1}^{3} \alpha_i \sigma_i\right).$$

Then

$$\operatorname{Tr}(\rho^2) = \frac{1}{4} \operatorname{Tr}\left(\left(1 + \sum_{i=1}^{3} \alpha_i^2\right) \sigma_0\right)$$

$$= \frac{1}{2}\left(1 + \sum_{i=1}^{3} \alpha_i^2\right)$$

ρ is pure $\Leftrightarrow$ $\frac{1}{2}\left(1 + \sum_{i=1}^{3} \alpha_i^2\right) = 1$

i.e. $\sum_{i=1}^{3} \alpha_i^2 = 1$.

Unitary freedom in the ensemble

A function $\eta : T \to \mathrm{Pos}(V)$ such that

$$\mathrm{Tr}\left(\sum_a \eta(a) \right) = 1$$

is called an ensemble of states.

The interpretation is that the system is in state

$$\rho_a = \frac{\eta(a)}{\mathrm{Tr}(\eta(a))} \quad \text{with probability}$$

$$p(a) = \mathrm{Tr}(\eta(a)) .$$

We will consider ensembles of pure states:

$$\rho = \sum_{i=1}^{n} p_i \, \tilde{v}_i \tilde{v}_i^{+} \in \mathrm{Conv}(P(V))$$

$$= \sum_i \sqrt{p_i}\, \tilde{v}_i \left(\sqrt{p_i}\, \tilde{v}_i \right)^{+}$$

$$= \sum_i v_i v_i^{+} .$$

This gives

$$\eta : \{1, \dots, n\} \longrightarrow \mathrm{Pos}(V)$$

$$\eta(i) = v_i v_i^{+} .$$

Thm: Let $\rho \in \mathrm{Den}(V)$. Then

$$\rho = \sum_i v_i v_i^+ = \sum_i w_i w_i^+$$

$$\iff \quad v_i = \sum_j U_{ij} w_j,$$

where $(U_{ij})_{ij}$ is a unitary matrix.

Proof: $(\Leftarrow)$

$$\sum_i w_i w_i^+ = \sum_i \sum_j U_{ij} v_j \sum_k \overline{U}_{ik} v_k^+$$

$$= \sum_{i,j,k} U_{ij} \underbrace{\overline{U}_{ik}}_{(U^+)_{ki}} v_j v_k^+$$

$$= \sum_{j,k} \underbrace{\sum_i (U^+)_{ki} U_{ij}}_{(U^+U)_{kj} = \delta_{kj}} v_j v_k^+$$

$$= \sum_j v_j v_k^+ .$$

$(\Rightarrow)$ Spectral decomposition $(d = \dim V)$

$$\rho = \sum_{k=1}^{d} \lambda_k \tilde{u}_k \tilde{u}_k^+$$

$$= \sum_{k \in \Sigma} u_k u_k^+$$

where $u_k = \sqrt{\lambda_k}\, \tilde{u}_k$ and $\Sigma = \{ 1 \le k \le d : \lambda_k \neq 0 \}$.

Claim: $v_i, w_i \in W = \mathrm{Span}\{u_k : k \in \Sigma\}$

If $v_i \notin W$ then $\langle u_k, v_i \rangle = 0 \quad \forall k \in \Sigma$.

Therefore

$$\langle v_i, \rho v_i \rangle = \sum_k \langle v_i, u_k u_k^+ v_i \rangle$$
$$= \sum_k |\underbrace{\langle u_k, v_i \rangle}_{0}|^2 = 0$$

$$\langle v_i, \rho v_i \rangle = \sum_{i'} \langle v_i, v_{i'} v_{i'}^+ v_i \rangle$$
$$= \sum_{i'} \underbrace{|\langle v_{i'}, v_i \rangle|^2}_{\langle v_{i'}, v_i \rangle = 0} = 0$$

But $\langle v_i, v_i \rangle \neq 0$ for v_i nonzero.

Similarly for w_i.

By the claim:

$$v_i = \sum_k c_{ik} u_k$$

We have

$$\rho = \sum_i v_i v_i^+$$
$$= \sum_i \sum_k c_{ik} u_k \sum_\ell \bar{c}_{i\ell} u_\ell^+$$
$$= \sum_{k,\ell} \left(\sum_i c_{ik} \bar{c}_{i\ell} \right) u_k u_\ell^+$$

On the other hand $\rho = \sum_k u_k u_k^+$.

Comparing the equations and using $\{u_k u_\ell^+\}$ is an orthonormal basis for $L(V)$:

$$\sum_i c_{ik} \bar{c}_{i\ell} = \delta_{k,\ell}$$

Define $A_{ik} = c_{ik}$ then:

$$\sum_i (A^+)_{\ell i} A_{ik} = \delta_{k,\ell} \quad \text{i.e,}$$

$$A^+ A = \mathbb{1}.$$

Similarly

$$w_i = \sum_k d_{ik} u_k \quad \text{where}$$

$$\sum_i d_{ik} \bar{d}_{i\ell} = \delta_{k,\ell}.$$

Define $B_{ik} = d_{ik}$ then:

$$B^+ B = \mathbb{1}.$$

This implies

$$\begin{aligned}
\sum_j (B^+)_{kj} w_j &= \sum_j (B^+)_{kj} \sum_\ell B_{j\ell} u_\ell \\
&= \sum_\ell \sum_j (B^+)_{kj} B_{j\ell} u_\ell \\
&= \sum_\ell (B^+ B)_{k\ell} u_\ell \\
&= u_k.
\end{aligned}$$

We have

$$
\begin{aligned}
x_i &= \sum_k A_{ik} u_k \\
&= \sum_k A_{ik} \sum_j (B^+)_{kj} w_j \\
&= \sum_j \Big(\sum_k A_{ik} (B^+)_{kj} \Big) w_j
\end{aligned}
$$

Letting $U = AB^+$ finishes the proof $\boxtimes$.

<u>HW</u>: $$
\begin{aligned}
U^+ U &= (AB^+)^+ A B^+ \\
&= B A^+ A B^+ \\
&= B B^+ = \mathbb{1} \iff B^+ B = \mathbb{1}.
\end{aligned}
$$

Partial trace

The partial trace

$$\mathrm{Tr}_V : L(V \otimes W) \to L(W)$$

is the unique linear operator that satisfies

$$\mathrm{Tr}_V(A \otimes B) = \mathrm{Tr}(A)\, B.$$

Equivalently $\mathrm{Tr}_V = \mathrm{Tr} \otimes \mathbb{1}_{L(W)}$

Similarly we can define

$$\mathrm{Tr}_W : L(V \otimes W) \to L(V).$$

Lem: $\mathrm{Tr}_W\big(\mathrm{Tr}_V(A)\big) = \mathrm{Tr}_V(\mathrm{Tr}_W(A))$

$$= \mathrm{Tr}(A).$$

proof: $L(V \otimes W)$ is the span of $(v_1 \otimes w_1)(v_2 \otimes w_2)^+$ where $v_i \in V$ and $w_i \in W$.

$$\mathrm{Tr}_W \mathrm{Tr}_V\big((v_1 \otimes w_1)(v_2 \otimes w_2)^+\big)$$

$$= \mathrm{Tr}_W \mathrm{Tr}_V(v_1 v_2^+ \otimes w_1 w_2^+)$$

$$= \mathrm{Tr}_W(\underbrace{\mathrm{Tr}(v_1 v_2^+)}_{\langle v_2, v_1\rangle}\, w_1 w_2^+)$$

$$= \langle v_2, v_1 \rangle \, \mathrm{Tr}\,(\underbrace{w_1 w_2^\dagger}_{\langle w_2, w_1 \rangle})$$

$$= \langle v_2 \otimes w_2 \,,\, v_1 \otimes w_1 \rangle$$

Similarly for $\mathrm{Tr}_V \mathrm{T}_W$. $\boxed{\checkmark}$

Lem: $\mathrm{Tr}_V^\dagger : L(W) \longrightarrow L(V \otimes W)$
is given by

$$\mathrm{Tr}_V^\dagger(A) = \mathbb{1}_V \otimes A.$$

Proof: We have

$$\langle B, \mathrm{Tr}_V A \rangle = \langle B, \mathrm{Tr} \otimes \mathbb{1}_{L(W)} A \rangle$$

$$= \langle (\mathrm{Tr} \otimes \mathbb{1}_{L(W)})^\dagger B, A \rangle$$

$$= \langle \mathrm{Tr}^\dagger \otimes \mathbb{1}_{L(W)} \underset{\uparrow}{B}, A \rangle$$

$1 \otimes B \in \mathbb{C} \otimes L(W)$

$$= \langle \underbrace{\mathrm{Tr}^\dagger(1)}_{\mathbb{1}_V} \otimes B, A \rangle$$

Therefore $\mathrm{Tr}_V^\dagger(B) = \mathbb{1}_V \otimes B$ $\boxed{\checkmark}$

Pro: $e \in \mathrm{Den}(V \otimes W)$ then $\mathrm{Tr}_V(e) \in \mathrm{Den}(W)$.
Similarly $\mathrm{Tr}_W(e) \in \mathrm{Den}(V)$.

proof: Let $e^{(W)} = \mathrm{Tr}_V(e)$.

Trace condition:

$$\mathrm{Tr}_W(e^{(W)}) = \mathrm{Tr}_W(\mathrm{Tr}_V(e))$$
$$= \mathrm{Tr}(e) = 1$$

Positivity condition:

For $Q \in \mathrm{Pos}(W)$ we have

$$\langle Q, \mathrm{Tr}_V(e)\rangle = \langle \mathbb{1}_V \otimes Q, e\rangle$$
$$\geq 0$$

since $\underbrace{\mathbb{1}_V \otimes Q}$ is positive.

$P \in \mathrm{Pos}(V)$, $Q \in \mathrm{Pos}(W) \Rightarrow$
$P \otimes Q \in \mathrm{Pos}(V \otimes W)$:

$P = A^{+}A$ & $Q = B^{+}B$ then

$$P \otimes Q = A^{+}A \otimes B^{+}B$$
$$= (A^{+} \otimes B^{+})(A \otimes B)$$
$$= (A \otimes B)^{+}(A \otimes B)$$

$\boxtimes$

Schmidt decomposition

Let $u \in V \otimes W$ be a unit vector.
There exists orthonormal sets

$$\{ v_i \in V \}_i \quad \text{and} \quad \{ w_i \in W \}_i$$

such that

$$u = \sum_i \lambda_i \, v_i \otimes w_i$$

see next one

where $\lambda_i \in \mathbb{R}_{\geq 0}$ and $\sum_i \lambda_i^2 = 1$.

proof: Let $\{ v_i' \}_{i=1}^n$ and $\{ w_j' \}_{j=1}^m$ be orthonormal basis for V and W.
We have

$$u = \sum_{i,j} c_{ij} \, v_i' \otimes w_j' . \qquad (*)$$

Let A be the square matrix:

$$A_{ij} = \begin{bmatrix} c_{11} & \cdots & c_{1m} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{n1} & & c_{nm} & 0 & \cdots & 0 \end{bmatrix} \quad \text{if } m \leq n$$

or

$$A_{ij} = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nm} \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{if } m > n$$

Consider the singular decomposition:

$$A = U D V \quad \text{where } D \text{ diagonal.}$$

Therefore

$$A_{ij} = \sum_k U_{ik} D_{kk} V_{kj} .$$

Substituting this in $(*)$:

$$u = \sum_{i,j,k} U_{ik} D_{kk} V_{kj} \; v_i' \otimes w_j' .$$

Let us define:

$$v_k = \sum_i U_{ik} v_i'$$

$$w_k = \sum_j V_{kj} w_j'$$

$$\lambda_i = D_{ii} .$$

Then

$$u = \sum_k D_{kk} \left(\sum_i U_{ik} v_i' \right) \otimes \left(\sum_j V_{kj} w_j' \right)$$

$$= \sum_k \lambda_k v_k \otimes w_k .$$

It remains to check that $\{v_i\}_i$ and $\{w_i\}_i$ are orthonormal:

$$\langle v_k, v_\ell \rangle = \langle \sum_i u_{ik} v_i', \sum_j u_{j\ell} v_j' \rangle$$

$$= \sum_{i,j} \overline{u_{ik}}\, u_{j\ell} \underbrace{\langle v_i', v_j' \rangle}_{\delta_{i,j}}$$

$$= \sum_i \overline{u_{ik}}\, u_{i\ell}$$

$$= (u^\dagger u)_{k\ell}$$

$$= \mathbb{1}_{k\ell} = \delta_{k\ell}.$$

Similarly for $\{w_i\}_i$ $\boxtimes$

<u>Jargon</u>:

λ_i : Schmidt coefficients

$\{v_i\}_i$ & $\{w_i\}_i$: Schmidt basis

$|\{i : \lambda_i \neq 0\}|$: Schmidt number of u.

<u>Cor</u>: Let $\rho = uu^\dagger \in P(V \otimes W)$.

Then spectral decomposition of the partial traces are given by

$$\rho^{(V)} = \sum_i \lambda_i^2 \, w_i w_i^\dagger$$

$$\rho^{(W)} = \sum_i \lambda_i^2 \, v_i v_i^\dagger$$

eigenvalues

$\rho \in \mathrm{Den}(V) \Rightarrow \sum_i \lambda_i = 1.$

Proof Schmidt decomposition:

$$u = \sum_i \lambda_i \, v_i \otimes w_i$$

Then $\mathrm{Tr}_V(\rho) = \mathrm{Tr}_V(uu^\dagger)$

$$= \sum_{i,j} \mathrm{Tr}_V\left(\lambda_i \lambda_j \, v_i \otimes w_i \, (v_j \otimes w_j)^\dagger\right)$$

$$= \sum_{i,j} \lambda_i \lambda_j \underbrace{\langle v_j , v_i \rangle}_{\delta_{i,j}} \, w_i w_j^\dagger$$

$$= \sum_i \lambda_i^2 \, w_i w_i^\dagger$$

Similarly for $\rho^{(w)}$ $\square$

Ex: Consider $|\phi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$

If $\rho = |\phi_{00}\rangle\langle\phi_{00}|$ then

$$\rho^{(v)} = \rho^{(w)} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$= \frac{1}{2}\mathbb{1}.$$

Def: A pure state $\rho \in P(V \otimes W)$ is called maximally entangled if

$$\rho^{(V)} = \frac{1}{\dim(W)} \mathbb{1}_W$$ completely mixed state

(or equivalently

$$\rho^{(W)} = \frac{1}{\dim(V)} \mathbb{1}_V \ .)$$

Cor: Let $\rho = uu^{+} \in P(V \otimes W)$.

The following are equivalent:

1) Schmidt number of u is 1

2) $\rho^{(V)}$ & $\rho^{(W)}$ pure.

3) ρ is a product state.

proof: HW.

Ex: Schmidt number of $|\phi_{00}\rangle$ is 2.

We know that it is not a product state.

Purification

Let $\rho \in \mathrm{Den}(V)$. There exists a Hilbert space W and a pure state $u \in P(V \otimes W)$ such that

$$\rho = \mathrm{Tr}_W (u u^+).$$

Proof: Spectral decomposition:

$$\rho = \sum_{i=1}^{d} p_i \, v_i v_i^+ , \quad d = \dim V.$$

Let $W = \mathbb{C}^d$.

Define

$$u = \sum_i \sqrt{p_i} \, \underbrace{v_i \otimes e_i}_{\text{Schmidt basis}}$$

(canonical orthonormal basis of $\mathbb{C}^d$.

Then

$$\mathrm{Tr}_W (u u^+) = \sum_{i,j} \sqrt{p_i} \sqrt{p_j} \, \mathrm{Tr}_W (v_i v_j^+ \otimes e_i e_j^+)$$

$$= \sum_i p_i \, v_i v_i^+ = \rho$$

$\boxtimes$

Unitary equivalence of purifications

Consider the linear isometry

$$\phi : L(\mathbb{C}^n) \longrightarrow \mathbb{C}^n \otimes \mathbb{C}^n$$

defined by

$$\phi(e_i e_j^\dagger) = e_i \otimes e_j .$$

Lemma: Let $A, B \in L(\mathbb{C}^n)$:

$$A \otimes B \; \phi(C) = \phi(A C B^T)$$

Proof: We can write

$$C = \sum_{ij} c_{ij} \, e_i e_j^\dagger$$

It suffices to show this for $C = e_i e_j^\dagger$:

$$\begin{aligned} A \otimes B \; \phi(e_i e_j^\dagger) &= (A \otimes B)\, e_i \otimes e_j \\ &= \underbrace{A e_i}_{\sum_k A_{ki} e_k} \otimes \underbrace{B e_j}_{\sum_\ell B_{\ell j} e_\ell} \\ &= \sum_{k,\ell} A_{ki} B_{\ell j} \, e_k \otimes e_\ell \end{aligned}$$

$$\begin{aligned} \phi(A e_i \underbrace{e_j^\dagger B^T}_{(\bar{B} e_j)^\dagger}) &= \phi\Big(\sum_k A_{ki} e_k \Big(\sum_\ell \bar{B}_{\ell j} e_\ell \Big)^\dagger \Big) \\ &= \sum_{k,\ell} A_{ki} B_{\ell j} \, e_k \otimes e_\ell \end{aligned}$$

$\square$

<u>Thm</u>: If u_1 & $u_2 \in V \otimes W$ satisfy

$$\mathrm{Tr}_W(u_1 u_1^+) = \mathrm{Tr}_W(u_2 u_2^+)$$

then there exists $U \in U(W)$ such that

$$u_2 = (\mathbb{1}_V \otimes U)\, u_1 .$$

<u>Proof</u>: Let $\{v_i\}_i$ & $\{w_j\}_j$ be orthonormal bases for V & W.

Then

$$u_1 = \sum_{i,j} \alpha_{ij}\, v_i \otimes w_j$$

$$u_2 = \sum_{i,j} \beta_{ij}\, v_i \otimes w_j .$$

Compute the partial traces:

$$\mathrm{Tr}_W(u_1 u_1^+) = \sum_i \alpha_{ii} \overline{\alpha_{ii}}\, v_i v_i^+$$

$$\mathrm{Tr}_W(u_2 u_2^+) = \sum_i \beta_{ii} \overline{\beta_{ii}}\, v_i v_i^+ .$$

We will reinterpret partial traces:

Let

$$A_1 = \sum_{i,j} \alpha_{ij}\, v_i w_j^+$$

$$A_2 = \sum_{i,j} \beta_{ij}\, v_i w_j^+$$

Then $\mathrm{Tr}_W(u_k \bar{u}_k) = A_k A_k^+ \qquad k = 1, 2 .$

We will perform singular value decomposition:

a) Spectral decomposition:

$$\rho = \sum_i \lambda_i \, x_i x_i^+$$

b) Then

$$A_k = \sqrt{\rho}\, U_k \qquad k=1,2.$$

Define

$$V = U_1^+ U_2 .$$

Then

$$A_1 V = (\sqrt{\rho}\, U_1)(U_1^+ U_2)$$
$$= \sqrt{\rho}\, U_2 = A_2 .$$

Let $U = V^T$. Then

$$(\mathbb{1}_V \otimes U)\, u_1 = (\mathbb{1}_V \otimes V^T)\, u_1$$
$$= (\mathbb{1}_V \otimes V^T)\, \phi(A_1)$$
$$= \phi(A_1 V) \qquad \text{(Lemma)}$$
$$= \phi(A_2)$$
$$= u_2$$

$\boxtimes$

Partial trace and measurements

Consider a system with two components with Hilbert spaces V and W.
Let $\rho \in \mathrm{Den}(V \otimes W)$.

Let

$$\Pi : \Sigma \longrightarrow \mathrm{Proj}(W)$$

be a projective measurement.
This can be extended to a projective meas:

$$\tilde{\Pi} : \Sigma \longrightarrow \mathrm{Proj}(V \otimes W)$$

$$\tilde{\Pi}_a = \mathbb{1}_V \otimes \Pi_a .$$

Then

$$\begin{aligned} p(a) &= \mathrm{Tr}(\tilde{\Pi}_a \rho) \\ &= \langle \mathbb{1}_V \otimes \Pi_a, \rho \rangle \\ &= \langle \Pi_a, \underbrace{\mathrm{Tr}_V \rho}_{\rho^{(W)}} \rangle \end{aligned}$$

Instead of measuring $\tilde{\Pi}$ on ρ we can measure Π on $\rho^{(W)}$.

Bell inequalities

A probability distribution on Σ is a function

$$p : \Sigma \to \mathbb{R}_{\geq 0} \quad \text{such that} \quad \sum_{a \in \Sigma} p(a) = 1.$$

Given $\rho \in \mathrm{Den}(V)$ and a POVM

$P : \Sigma \to \mathrm{Pos}(V)$ we obtain a prob. dist:

$$p : \Sigma \to \mathbb{R}_{\geq 0}$$

$$p(a) = \mathrm{Tr}(\rho P_a).$$

Let $A \in \mathrm{Herm}(V)$ and $B \in \mathrm{Herm}(W)$ with eigenvalues ± 1.

Simultaneous diagonalization of $A \otimes \mathbb{1}_W$ and $\mathbb{1}_V \otimes B$:

$$\Pi_{AB} : \Sigma \to \mathrm{Proj}(V \otimes W)$$

where $\Sigma = \{00, 01, 10, 11\}$ and

$$\Pi_{AB}^{ab} = \Pi_A^a \otimes \Pi_B^b.$$

Recall that

$$A = \Pi_A^0 - \Pi_A^1$$

$$B = \Pi_B^0 - \Pi_B^1.$$

We will also consider

$$\Pi_{A\otimes B} : \{0,1\} \longrightarrow \mathrm{Proj}(\mathbb{C}^2)$$

obtained from the spectral decomposition of $A \otimes B$.

Lemma: Spectral decomposition of $A\otimes B$ is given by

$$A \otimes B = \Pi^0_{A\otimes B} - \Pi^1_{A\otimes B}$$

where

$$\Pi^a_{A\otimes B} = \Pi^{0a}_{AB} + \Pi^{1(a+1)}_{AB} .$$

Proof: We have

$$A\otimes B = (\Pi^0_A - \Pi^1_A) \otimes (\Pi^0_B - \Pi^1_B)$$

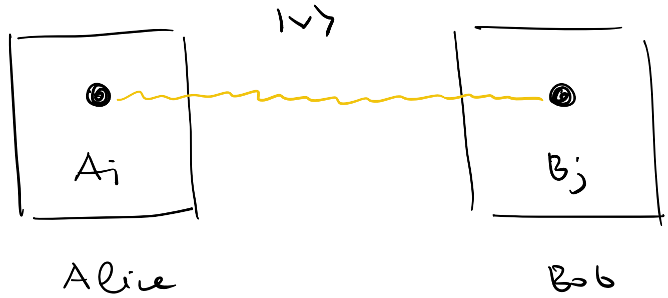
$$= \Pi^{00}_{AB} + \Pi^{11}_{AB} - (\Pi^{01}_{AB} + \Pi^{10}_{AB})$$

$\blacksquare$

Cor: $\quad P_{A\otimes B}(a) = P_{AB}(0a) + P_{AB}(1(a+1))$

HW: Verify this.

Clauser – Horne – Shimony – Holt (CHSH) inequalities



$i, j \in \{0, 1\}$

HW: Alice measures in $\hat{z}$ & Bob in $\hat{\alpha}$.
Find the probabilities.

For $ij \in \Gamma = \{00, 01, 10, 11\}$ we have a projective measurement

$$\Pi_{A_i B_j} : \Sigma_{ij} \longrightarrow \text{Proj}(V \otimes W)$$

where $\Sigma_{ij} = \{00, 01, 10, 11\}$ and $V = W = \mathbb{C}^2$.

Let $\rho \in \text{Den}(V \otimes W)$.

Born rule gives a distribution

$$p_{A_i B_j} : \Sigma_{ij} \longrightarrow \mathbb{R}_{\geq 0}$$

$$p_{A_i B_j}(ab) = \text{Tr}(\rho\, \Pi^{ab}_{A_i B_j})$$

for each $ij \in \Gamma$.

Joint probability distribution

A joint probability distribution for

$$\{ p_{A_i B_j} \}_{i,j}$$

is a probability distribution

$$p : \Sigma \longrightarrow \mathbb{R}_{\geq 0}$$

where $\Sigma = \{ a_0 b_0 a_1 b_1 : a_i , b_j \in \{0,1\} \}$

such that

$$\sum_{a_{\bar{i}}, b_{\bar{j}}} p(a_0 b_0 a_1 b_1) = p_{A_i B_j}(a_i b_j)$$

Notation: $\bar{i} = \mathrm{NOT}(i) = \begin{cases} 0 & i=1 \\ 1 & i=0 . \end{cases}$

Correlation function

We have

$$\Pi_{A_i \otimes A_j} : \{0,1\} \longrightarrow \mathrm{Proj}(\mathbb{C}^2) .$$

and the corresponding distribution:

$$p_{A_i \otimes B_j} : \{0,1\} \longrightarrow \mathbb{R}_{\geq 0} .$$

The correlation function is defined by

$$C : \Gamma \longrightarrow \mathbb{R}$$

$$C_{ij} = p_{A_i \otimes B_j}(0) - p_{A_i \otimes B_j}(1) .$$

Theorem: There exists a joint probability distribution $\iff$ the CHSH inequalities are satisfied:

$$-2 \le C_{00} + C_{01} + C_{10} - C_{11} \le 2$$
$$-2 \le C_{00} + C_{01} - C_{10} + C_{11} \le 2$$
$$-2 \le C_{00} - C_{01} + C_{10} + C_{11} \le 2$$
$$-2 \le -C_{00} + C_{01} + C_{10} + C_{11} \le 2.$$

proof: We will only prove $(\Rightarrow)$.

We have

$$C_{ij} = p_{A_i \otimes B_j}(0) - p_{A_i \otimes B_j}(1)$$
$$= p_{A_i B_j}(00) + p_{A_i B_j}(11) - \left(p_{A_i B_j}(01) + p_{A_i B_j}(10) \right)$$
$$= \sum_{\substack{a_0, b_0 \\ a_1, b_1}} (-1)^{a_i + b_j} p(a_0 b_0 a_1 b_1)$$

We can rewrite the CHSH ineq:

$$-2 \le \sum_{i,j} (-1)^{(i+a)\cdot(j+b)} C_{ij} \le 2$$

where $a, b \in \{0, 1\}$.

Then

$$\sum_{i,j} (-1)^{(i+a)\cdot(j+b)} \; C_{ij}$$

$$= \sum_{i,j} (-1)^{(i+a)\cdot(j+b)} \sum_{\substack{a_0,b_0\\ a_1,b_1}} (-1)^{a_i+b_j} \; p(a_0 b_0 a_1 b_1)$$

$$= \sum_{\substack{a_0,b_0\\ a_1 b_1}} p(a_0 b_0 a_1 b_1) \sum_{i,j} (-1)^{(i+a)\cdot(j+b)+a_i+b_j}$$

The CHSH inequalities follow from

$$-2 \leq \underbrace{\sum_{i,j} (-1)^{(i+a)\cdot(j+b)+a_i\cdot b_j}}_{\pm(1+1+1-1)} \leq 2 .$$

$\square$

Quantum mechanical correlations

$$C_{ij} = p_{A_i \otimes B_j}(0) - p_{A_i \otimes B_j}(1)$$

$$= \mathrm{Tr}(\Pi^0_{A_i \otimes B_j} \rho) - \mathrm{Tr}(\Pi^1_{A_i \otimes B_j} \rho)$$

$$= \mathrm{Tr}(A_i \otimes B_j \rho)$$

We will compute this for

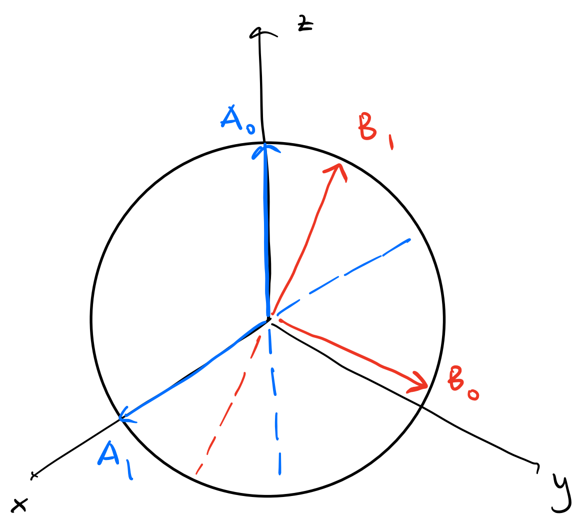
$$\rho = |v\rangle\langle v| \quad \text{where}$$

$$|v\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

and

$$A_0 = Z, \quad B_0 = (-Z - X)/\sqrt{2}$$

$$A_1 = X, \quad B_1 = (Z - X)/\sqrt{2}$$



We have

$$C_{00} = \mathrm{Tr}(\rho \, A_0 \otimes B_0)$$

$$= \langle v | Z \otimes \frac{-Z-X}{\sqrt{2}} | v \rangle$$

$$= \left(\frac{\langle 01| + \langle 10|}{\sqrt{2}} \right) \left(\frac{|01\rangle + |10\rangle}{2} \right) = \frac{1}{\sqrt{2}}$$

Similarly $C_{01} = C_{10} = -C_{11} = 1/\sqrt{2}$. (HW)

Then

$$\sum_{i,j} (-1)^{i \cdot j} C_{ij} = \frac{1}{\sqrt{2}} (1 + 1 + 1 - (-1)) = 2\sqrt{2}$$

The ~~first~~ CHSH inequality is violated.

Therefore $\{ P_{A_i B_j} \}_{i,j}$ does not admit a joint probability distribution.

Interpretation: Observables do not have predetermined outcomes before the measurement.